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Bifurcations of the von Kármán Equations with Robin Boundary Conditions

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Abstract—We study buckling of a rectangular plate with elastic support and restraint along two edges and simple support along the other two edges. This corresponds to a bifurcation analysis of the von Kármán equations with Robin boundary conditions. Of special interest is the phenomenon of mode jumping in deformation of the plate. We discretize the von Kármán equations with the finite difference methods. The solution curves branching from the first two simple bifurcation points, resulted from splitting of a double bifurcation point, are numerically traced with respect to the increase of the load by using the continuation methods. Our numerical results show that mode jumping depends on length of the plate and stiffness of the elastic support.

Keywords—von Kármán equations, Robin boundary conditions, Mode jumping, Bifurcation, Continuation methods.

1. INTRODUCTION

Deformation of a full three-dimensional elastic plate under compression is modeled approximately by the von Kármán equations

$$\begin{aligned} \Delta^2 f + \frac{1}{2}[w,w] &= 0, \\ \Delta^2 w + \lambda \frac{\partial^2 w}{\partial x^2} - [f,w] &= 0, \end{aligned} \qquad \text{in } \Omega. \tag{1.1}$$

Here Ω represents the shape of the plate in its flat state, f is the Airy stress function describing the averaged stress over the thickness of the plate, $w(x,y)$ is the deformation of the plate under

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action of the external load λ, Δ^2 is the biharmonic operator in the plane, and the bracket operator $[\cdot, \cdot]$ is defined by

$$[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}.$$

The von Kármán equations are derived as leading terms in the asymptotic expansion of deformation of the three-dimensional plate, in which the load is coupled with the thickness of the plate, see e.g., [1, Chapter 14; 2]. The simply supported boundary conditions

$$w = \Delta w = 0, \tag{1.2a}$$
$$f = \Delta f = 0, \qquad \text{on } \partial\Omega \tag{1.2b}$$

are often imposed on the von Kármán equations for simplifying mathematical analysis, although they are hard to realize experimentally, see e.g., [3,4]. For the Airy stress f Schaeffer and Golubitsky showed in [3] that physically the boundary conditions

$$\frac{\partial f}{\partial n} = \frac{\partial}{\partial n}(\Delta f) = 0, \qquad \text{on } \partial\Omega \tag{1.3}$$

are more appropriate than (1.2b). Here and below $\frac{\partial}{\partial n}$ denotes the exterior normal derivative. Solving f formally as a function of w from the equation

$$\Delta^2 f = -\frac{1}{2}[w, w]$$

under the boundary conditions (1.3) and the restriction $\int_\Omega f dx = 0$, we obtain

$$f = -\frac{1}{2}\Delta_N^{-2}[w, w],$$

where Δ_N^{-1} denotes the inverse operator of the Laplacian with the homogeneous Neumann boundary conditions (1.3). Consequently, (1.1) is equivalent to the semilinear elliptic equation

$$\Delta^2 w + \lambda\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left[\Delta_N^{-2}[w, w], w\right] = 0. \tag{1.4}$$

Similar conclusion holds for f with respect to the simply supported boundary conditions (1.2b). This indicates that different types of boundary conditions for f have no influence on the linear instability of the flat state of a plate.

The impact of the Airy stress f on deformation of the plate come in via secondary bifurcations, e.g., mode jumping, which is a remarkable characteristic of experimental studies of the post-buckling of plate. Holder and Schaeffer [5] and Schaeffer and Golubitsky [3] interpreted the phenomenon of mode jumping in terms of secondary bifurcations which occur when the primary solution branches lose stability through further bifurcations. More precisely, it is known that a double bifurcation can be split into two simple bifurcations by perturbation. For the von Kármán equations if the solution curves branching from the first two simple bifurcation points, which correspond to the splitting of the first double bifurcation point, are connected by a secondary solution branch, then we say that mode jumping occurs, see [6] for detailed discussions on spring models. Mode jumping has been predicted by Schaeffer and Golubitsky [3] for the rectangular plate

$$\Omega := [0, \ell] \times [0, 1] \tag{1.5}$$

with boundary conditions

$$\begin{aligned} w = \frac{\partial w}{\partial n} = 0, &\qquad \text{for } x = 0, \ell, \\ w = \Delta w = 0, &\qquad \text{for } y = 0, 1, \end{aligned} \tag{1.6}$$

for w and either (1.2b) or (1.3) for f, see also [5]. The theoretical prediction has been confirmed numerically by Chien *et al.* [7]. The boundary conditions (1.6) correspond to clamped ends ($x = 0, \ell$) and simply supported sides ($y = 0, 1$) of the rectangular plate. We wish to relax the clamped boundary conditions at ends of the plate and consider the Robin boundary conditions

$$\begin{aligned} g_0(\mu)\frac{\partial w}{\partial n} + g_1(\mu)\Delta w &= 0, && \text{for } x = 0, \ell, \\ \Delta w &= 0, && \text{for } y = 0, 1, \\ w &= 0, && \text{on } \partial\Omega, \\ \frac{\partial f}{\partial n} = \frac{\partial \Delta f}{\partial n} &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

where $g_i(\mu) \in \mathbf{R},\ i = 1, 2$ are smooth functions and satisfy

$$\begin{aligned} g_0(0) \cdot g_1(1) \cdot g_0(\mu) \cdot g_1(\mu) &\neq 0, && \text{for all } \mu \in (0, 1), \\ g_0(1) = g_1(0) &= 0. \end{aligned} \tag{1.8}$$

When $\mu = 1$ we have the simply supported boundary conditions (1.2a),(1.2b), while for $\mu = 0$ we obtain the partially clamped case (1.6). The boundary conditions (1.7) can be thought as a plate with partially fixed and elastic restrained edge, for example, an edge beam with torsional rigidity $g_1(\mu)/g_0(\mu)$, see [8, Chapter 1] and Figure 1.

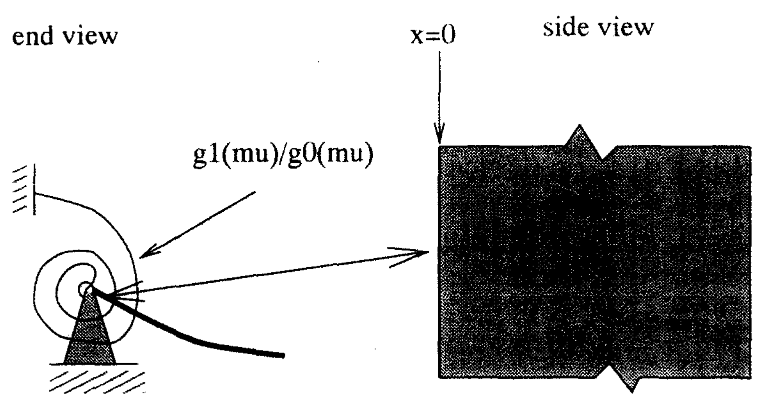

Figure 1. A rectangular plate with partially fixed and elastic restrained edge.

We are interested in mode jumping of the plate with the Robin boundary conditions (1.7), and dependence of buckling of the plate on the torsional rigidity. Physically the boundary conditions (1.7) are more realistic than (1.6). The price to be paid here, however, is that the detailed and beautiful local analysis of Golubitsky and Schaeffer becomes quite difficult to carry out. This is due to the fact that even the solutions of linearized problem are now only semiexlicit, see Section 2 for details. To this end, we resort to a global numerical study of this problem and investigate bifurcation scenario of the von Kármán equations by varying the homotopy parameter μ.

In this paper, we present some numerical bifurcation analysis of the von Kármán equations (1.1) with the Robin boundary conditions (1.7). In Section 2, we discuss the linear stability of the flat state of the plate. A brief bifurcation analysis of simple bifurcations is given in Section 3. Section 4 is devoted to the central difference approximation of the von Kármán equations with the Robin boundary conditions. Numerical results are reported in Section 5. Finally, some concluding remarks are given in Section 6.

2. LINEAR STABILITY ANALYSIS

To study stabilities of the flat state $w \equiv 0$, $f \equiv 0$ of the plate for $\lambda \in \mathbf{R}$, we examine the linearization of (1.4)

$$\Delta^2 w + \lambda \frac{\partial^2 w}{\partial x^2} = 0, \tag{2.1}$$

imposed with the Robin boundary conditions

$$\begin{aligned} w &= 0, && \text{on } \partial\Omega, \\ \Delta w &= 0, && \text{for } y = 0, 1, \\ -g_0(\mu)\frac{\partial w}{\partial x} + g_1(\mu)\Delta w &= 0, && \text{for } x = 0, \\ g_0(\mu)\frac{\partial w}{\partial x} + g_1(\mu)\Delta w &= 0, && \text{for } x = \ell, \end{aligned} \tag{2.2}$$

where $g_0(\mu)$, $g_1(\mu)$ satisfy condition (1.8). This is a generalized eigenvalue problem. For any nontrivial solution (w, λ) of (1.8), λ is called the eigenvalue and w is the corresponding eigenfunction. The spectrum of (2.1) reveals stability of the flat state of the plate.

In the space

$$C_0^4(\Omega) := \left\{ u \in C^4(\Omega) : u \text{ satisfies } (2.2) \right\},$$

the operator $\Delta^2 + \lambda \frac{\partial^2}{\partial x^2}$ is self-adjoint and positive definite for $\lambda \leq 0$ with respect to the $L^2(\Omega)$-norm. In fact, for all u, $v \in C_0^4(\Omega)$, and $\mu \neq 1$ we have

$$\begin{aligned} &\int_\Omega u \left(\Delta^2 v + \lambda \frac{\partial^2 v}{\partial x^2} \right) dx\,dy \\ &= \int_\Omega \Delta u \Delta v\, dx\,dy + \int_{\partial\Omega} \left(u \frac{\partial \Delta v}{\partial n} - \Delta v \frac{\partial u}{\partial n} \right) ds - \lambda \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx\,dy + \lambda \int_{\partial\Omega} u \frac{\partial v}{\partial x}\, ds \\ &= \int_\Omega \Delta u \Delta v\, dx\,dy - \int_{\partial\Omega} \Delta v \frac{\partial u}{\partial n}\, ds - \lambda \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx\,dy \\ &= \int_\Omega \Delta u \Delta v\, dx\,dy - \lambda \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx\,dy - \int_{x=0} \Delta v \frac{\partial u}{\partial n}\, ds - \int_{x=\ell} \Delta v \frac{\partial u}{\partial n}\, ds \\ &= \int_\Omega \Delta u \Delta v\, dx\,dy - \lambda \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\, dx\,dy + \frac{g_1(\mu)}{g_0(\mu)} \left(\int_{x=0} \Delta v \Delta u\, ds + \int_{x=\ell} \Delta v \Delta u\, ds \right). \end{aligned}$$

Since the last equality is symmetric with respect to u and v, the operator $\Delta^2 + \lambda \frac{\partial^2}{\partial x^2}$ is self-adjoint. Moreover, this operator is singular only if λ is equal to some eigenvalues of (2.1), which satisfies

$$\lambda = \frac{\left[\int_\Omega (\Delta w)^2\, dx\,dy + (g_1(\mu)/g_0(\mu)) \left(\int_{x=0} (\Delta w)^2\, ds + \int_{x=\ell} (\Delta w)^2\, ds \right) \right]}{\left[\int_\Omega \left(\frac{\partial w}{\partial x} \right)^2 dx\,dy \right]}$$

for the corresponding eigenfunction. This implies that all eigenvalues λ are positive. Bifurcation points of the von Kármán equations on the trivial solution curve are given by $(0, \lambda_0)$ with $\lambda_0 > 0$ as an eigenvalue of (2.1).

To study variations of the eigenvalues of (2.1) along the homotopy of boundary conditions (2.2), we apply the rule of separation of variables to (2.1), see also [3]. We consider the following form of solutions of (2.1):

$$w(x, y) = u(x) \cdot v(y) \not\equiv 0.$$

Substituting it into (2.1) yields

$$\frac{u^{(4)} + \lambda u''}{u} + 2\frac{u''v''}{uv} + \frac{v^{(4)}}{v} = 0. \tag{2.3}$$

Correspondingly, the boundary conditions (2.2) reduce to

$$\begin{aligned} v(0) &= v(1) = 0, \\ v''(0) &= v''(1) = 0, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} u(0) &= u(\ell) = 0, \\ -g_0(\mu)u'(0) + g_1(\mu)u''(0) &= 0, \\ g_0(\mu)u'(\ell) + g_1(\mu)u''(\ell) &= 0, \end{aligned} \tag{2.5}$$

respectively. We choose the basis $\{\sin m\pi y;\ m \in \mathbf{N}\}$ for the space of functions

$$\left\{v \in C^4[0,1]:\ v(0) = v(1) = v''(0) = v''(1) = 0\right\}.$$

Since the subspace $\{u(x)\sin m\pi y :\ m \in \mathbf{N}\}$ is invariant under the operator $\Delta^2 + \lambda\frac{\partial^2}{\partial x^2}$, choosing $v(y) = \sin m\pi y$ reduces (2.3) into a fourth-order linear differential equation for u,

$$u^{(4)} + \left(\lambda - 2m^2\pi^2\right)u'' + m^4\pi^4 u = 0. \tag{2.6}$$

The characteristic of this equation is

$$a^4 + \left(\lambda - 2m^2\pi^2\right)a^2 + (m\pi)^4 = 0,$$

which has four solutions

$$\begin{aligned} a_1 &:= \left[m^2\pi^2 - \frac{\lambda}{2} + \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right]^{1/2}, \qquad a_2 := -a_1, \\ a_3 &:= \left[m^2\pi^2 - \frac{\lambda}{2} - \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right]^{1/2}, \qquad a_4 := -a_3. \end{aligned} \tag{2.7}$$

Thereafter, we obtain the general solutions of (2.6)

$$u(x) = \sum_{i=1}^{4} \alpha_i e^{a_i x}, \tag{2.8}$$

where the constants $\alpha_i \in \mathbf{C},\ i = 1,2,3,4$ are determined by the boundary conditions (2.5). To ensure real solutions, we consider the following three cases:

$$\frac{\lambda}{4} - m^2\pi^2 < 0, \tag{2.9a}$$

$$\frac{\lambda}{4} - m^2\pi^2 = 0, \tag{2.9b}$$

$$\frac{\lambda}{4} - m^2\pi^2 > 0. \tag{2.9c}$$

2.1. Case $\lambda \in \left(0, 4m^2\pi^2\right)$

Since

$$\sqrt{a+bi} = \frac{1}{\sqrt{2}}\left[\left(\sqrt{a^2+b^2}+a\right)^{1/2} + \left(\sqrt{a^2+b^2}-a\right)^{1/2} i\right], \qquad \text{for all } a,b \in \mathbf{R},$$

the inequality $\operatorname{Re}\sqrt{a+bi}\cdot\operatorname{Im}\sqrt{a+bi} \neq 0$ holds for $b \neq 0$. If $\lambda \in (0, 4m^2\pi^2)$, then

$$\begin{aligned} a &:= \operatorname{Re}(a_1) = -\operatorname{Re}(a_2) = \operatorname{Re}(a_3) = -\operatorname{Re}(a_4) \neq 0, \\ b &:= \operatorname{Im}(a_1) = -\operatorname{Im}(a_2) = -\operatorname{Im}(a_3) = +\operatorname{Im}(a_4) \neq 0. \end{aligned} \tag{2.10}$$

Substituting (2.8) into (2.5) yields a linear system for the α_i, $i = 1,2,3,4$

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0,\\ \alpha_1 e^{a_1\ell} + \alpha_2 e^{a_2\ell} + \alpha_3 e^{a_3\ell} + \alpha_4 e^{a_4\ell} &= 0,\\ -g_0(\mu)\sum_{i=1}^4 a_i\alpha_i + g_1(\mu)\sum_{i=1}^4 a_i^2\alpha_i &= 0,\\ g_0(\mu)\sum_{i=1}^4 a_i\alpha_i e^{a_i\ell} + g_1(\mu)\sum_{i=1}^4 a_i^2\alpha_i e^{a_i\ell} &= 0. \end{aligned} \tag{2.11}$$

On the other hand, the function

$$u(x) = \sum_{i=1}^4 \alpha_i e^{a_i x} = e^{ax}\left(\alpha_1 e^{ibx} + \alpha_3 e^{-ibx}\right) + e^{-ax}\left(\alpha_2 e^{-ibx} + \alpha_4 e^{ibx}\right)$$

is real if and only if

$$\alpha_1 = \overline{\alpha_3}, \qquad \alpha_2 = \overline{\alpha_4}. \tag{2.12}$$

Together with (2.11) we obtain an over-determined system for α_i, $i = 1,2,3,4$, which only has trivial solution.

2.2. Case $\lambda = 4m^2\pi^2$

In this case, we derive from (2.7) that

$$a_1 = -a_2 = a_3 = -a_4 = m\pi \cdot i.$$

The solution $u(x)$ of (2.6) is of the form

$$u(x) = \alpha_1 \cos m\pi x + \alpha_2 \sin m\pi x..$$

Therefore, the boundary conditions (2.5) imply $\alpha_1 = 0$ and

$$\begin{aligned} \alpha_1 \cos m\ell\pi + \alpha_2 \sin m\ell\pi &= 0,\\ -g_0(\mu)\alpha_2 m\pi - g_1(\mu)\alpha_1 m^2\pi^2 &= 0,\\ g_0(\mu)m\pi(-\alpha_1 \sin m\ell\pi + \alpha_2\cos m\ell\pi) - g_1(\mu)m^2\pi^2(\alpha_1\cos m\ell\pi + \alpha_2 \sin m\ell\pi) &= 0. \end{aligned}$$

Hence, we have either

$$\alpha_1 = \alpha_2 = 0 \Longrightarrow u(x) \equiv 0, \qquad \text{(the trivial solution)}$$

or

$$\alpha_1 = 0, \qquad g_0(\mu) = 0, \qquad \ell \in \mathbf{N} \Longrightarrow u(x) = \alpha_2 \sin k\pi x,$$

which reduces to the case of simply supported boundary conditions.

2.3. Case $\lambda > 4m^2\pi^2$

In this case, we have

$$\begin{aligned} a_1 &= -a_2 = i\omega_1, & \omega_1 &:= \left(\frac{\lambda}{2} - m^2\pi^2 - \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right)^{1/2},\\ a_3 &= -a_4 = i\omega_2, & \omega_2 &:= \left(\frac{\lambda}{2} - m^2\pi^2 + \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right)^{1/2} > 0. \end{aligned} \tag{2.13}$$

For these pure imaginary values, we can write the solution $u(x)$ of (2.6) as

$$u(x) = \alpha_1 \cos\omega_1 x + \alpha_2 \sin\omega_1 x + \alpha_3 \cos\omega_2 x + \alpha_4 \sin\omega_2 x. \tag{2.14}$$

Thereafter, the boundary conditions (2.5) reduce to

$$\begin{aligned} \alpha_1 + \alpha_3 &= 0, \\ \alpha_1 \cos\omega_1\ell + \alpha_2 \sin\omega_1\ell + \alpha_3 \cos\omega_2\ell + \alpha_4 \sin\omega_2\ell &= 0, \\ -g_0(\mu)(\alpha_2\omega_1 + \alpha_4\omega_2) - g_1(\mu)\left(\omega_1^2\alpha_1 + \omega_2^2\alpha_3\right) &= 0, \\ g_0(\mu)\left(-\alpha_1\omega_1 \sin\omega_1\ell + \alpha_2\omega_1\cos\omega_1\ell - \alpha_3\omega_2\sin\omega_2\ell + \alpha_4\omega_2\cos\omega_2\ell\right) & \\ -g_1(\mu)\left(\alpha_1\omega_1^2\cos\omega_1\ell + \alpha_2\omega_1^2\sin\omega_1\ell + \alpha_3\omega_2^2\cos\omega_2\ell + \alpha_4\omega_2^2\sin\omega_2\ell\right) &= 0. \end{aligned} \tag{2.15}$$

REMARK. If $g_0 = 0$, then $\alpha_1 = \alpha_3 = 0$, and

$$u(x) = \alpha_2 \sin\omega_1 x + \alpha_4 \sin\omega_2 x. \tag{2.16}$$

If $g_1 = 0$, then $\alpha_3 = -\alpha_1$, $\alpha_4 = -\alpha_2\omega_1/\omega_2$. Thereafter, the solution $u(x)$ in (2.8) can be written as

$$u(x) = \alpha_1\left(\cos\omega_1 x - \cos\omega_2 x\right) + \alpha_2\left(\sin\omega_1 x - \frac{\omega_1}{\omega_2}\sin\omega_2 x\right).$$

These coincide with the results in [3].

In the sequel, we restrict the discussion to $g_0 \cdot g_1 \neq 0$. It follows from (2.15) that

$$\begin{aligned} \alpha_3 &= -\alpha_1, \\ \alpha_4 &= \frac{1}{\omega_2}\left[-\frac{g_1(\mu)}{g_0(\mu)}\left(\omega_1^2 - \omega_2^2\right)\alpha_1 - \omega_1\alpha_2\right]. \end{aligned}$$

Substituting them into (2.14) and multiplying with the factor $g_0(\mu)\omega_2$ yields

$$\begin{aligned} u(x) = {} & \alpha_1\left(g_0\omega_2(\cos\omega_1 x - \cos\omega_2 x) - g_1\left(\omega_1^2 - \omega_2^2\right)\sin\omega_2 x\right) \\ & + \alpha_2 g_0\left(\omega_2\sin\omega_1 x - \omega_1\sin\omega_2 x\right). \end{aligned} \tag{2.17}$$

Thereafter, the boundary conditions (2.5) reduce to a system for α_1, α_2, namely,

$$\begin{aligned} m_{11}\alpha_1 + m_{12}\alpha_2 &= 0, \\ m_{21}\alpha_1 + m_{22}\alpha_2 &= 0, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} m_{11} &= g_0\omega_2\left(\cos\omega_1\ell - \cos\omega_2\ell\right) - g_1\left(\omega_1^2 - \omega_2^2\right)\sin\omega_2\ell, \\ m_{12} &= g_0\left(\omega_2\sin\omega_1\ell - \omega_1\sin\omega_2\ell\right), \\ m_{21} &= g_0^2\left[-\omega_1\omega_2\sin\omega_1\ell + \omega_2^2\sin\omega_2\ell\right] + g_1^2\omega_2^2\left(\omega_1^2 - \omega_2^2\right)\sin\omega_2\ell \\ &\quad + g_0 g_1\left[-\omega_1^2\omega_2\cos\omega_1\ell + \omega_2^3\cos\omega_2\ell - \omega_2\left(\omega_1^2 - \omega_2^2\right)\cos\omega_2\ell\right], \\ m_{22} &= g_0^2\omega_2\omega_1\left(\cos\omega_1\ell - \cos\omega_2\ell\right) - g_0 g_1\omega_2\omega_1\left(\omega_1\sin\omega_1\ell - \omega_2\sin\omega_2\ell\right). \end{aligned} \tag{2.19}$$

The homogeneous system (2.18) has nontrivial solutions if and only if the determinant of its coefficient matrix vanishes, i.e.,

$$m_{11}m_{22} - m_{12}m_{21} = 0.$$

This gives an equation for λ with μ, m, and ℓ as parameters, i.e.,

$$\begin{aligned} q(\lambda, \mu, \ell, m) := {} & g_0^2\left[2\omega_1\omega_2\left(1 - \cos\omega_1\ell\cos\omega_2\ell\right) - \left(\omega_1^2 + \omega_2^2\right)\sin\omega_1\ell\sin\omega_2\ell\right] \\ & + g_0 g_1\left[2\left(\omega_2^2 - \omega_1^2\right)\left(\omega_1\sin\omega_2\ell\cos\omega_1\ell - \omega_2\sin\omega_1\ell\cos\omega_2\ell\right)\right] \\ & + g_1^2\left(\omega_2^2 - \omega_1^2\right)^2\sin\omega_1\ell\sin\omega_2\ell = 0. \end{aligned} \tag{2.20}$$

Recall that ω_1, ω_2 are defined in (2.13). We calculate solutions of equation (2.20) by using the numerical continuation methods with respect to the parameter ℓ or μ. For each solution $\lambda_m(\ell,\mu)$ of (2.20), by (2.17) equation (2.6) generically has a one-dimensional solution space spanned by

$$u_m(x) = \alpha_1\psi_1(x) + \alpha_2\psi_2(x). \tag{2.21}$$

Here α_1, α_2 satisfy (2.18), and

$$\begin{aligned} \psi_1(x) &:= g_0(\mu)\omega_2(\cos\omega_1 x - \cos\omega_2 x) - g_1(\mu)\left(\omega_1^2 - \omega_2^2\right)\sin\omega_2 x, \\ \psi_2(x) &:= g_0(\mu)(\omega_2 \sin\omega_1 x - \omega_1 \sin\omega_2 x) \end{aligned} \tag{2.22}$$

are implicit functions of the parameters ℓ and μ, as well as the mode number m in the y-direction, see also (2.17).

We conclude the discussion above as follows.

THEOREM 1. *Any eigenpair of the linear eigenvalue problem (2.1) with boundary conditions (2.2) is of the form $(\lambda_m, w_m(x,y))$, $m \in \mathbf{N}$, where λ_m is greater than $4m^2\pi^2$ and satisfies equation (2.20), and $w_m(x,y)$ is given as*

$$w_m(x,y) = (\alpha_1\psi_1(x) + \alpha_2\psi_2(x))\sin m\pi y$$

with ψ_1, ψ_2 in (2.22) and (α_1, α_2) satisfying the linear system (2.18).

Some special solutions of (2.20) are related directly to the classical results.

(a) Simply supported boundary conditions ($\mu = 1$).
Since $g_0(1) = 0$, the equation $q(\lambda, 1, \ell, m) = 0$ reduces to

$$-g_1^2(1)\left(\lambda - 4m^2\pi^2\right)\sin\left(\sqrt{\lambda/2 - m^2\pi^2 - \sqrt{\lambda\left(\lambda/4 - m^2\pi^2\right)}}\,\ell\right) \sin\left(\sqrt{\lambda/2 - m^2\pi^2 + \sqrt{\lambda(\lambda/4 - m^2\pi^2)}}\,\ell\right) = 0,$$

which can be also derived directly from (2.16) and (2.5). Thus by $\lambda > 4m^2\pi^2$, we obtain

$$\left(\frac{\lambda}{2} - m^2\pi^2 \pm \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right)^{1/2} \ell = k\pi, \qquad \text{for some } k \in \mathbf{N}.$$

Therefore,

$$\lambda = \left(m^2 + \frac{k^2}{\ell^2}\right)^2 \frac{\ell^2}{k^2}\pi^2. \tag{2.23}$$

(b) Clamped edges ($\mu = 0$).

We obtain from $g_1(0) = 0$, $g_0(0) \neq 0$, and $q(\lambda, 0, \ell, m) = 0$ that

$$2m^2\pi^2\left(1 - \cos\omega_1\ell\cos\omega_2\ell\right) - \left(\lambda - 2m^2\pi^2\right)\sin\omega_1\ell\sin\omega_2\ell = 0. \tag{2.24}$$

Now, if we choose $\omega_1\ell = k\pi$, $\omega_2\ell = (k+2n)\pi$, i.e.,

$$\left(\frac{\lambda}{2} - m^2\pi^2 - \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right)^{1/2} \ell = k\pi, \tag{2.25}$$

$$\left(\frac{\lambda}{2} - m^2\pi^2 + \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right)^{1/2} \ell = (k+2n)\pi \tag{2.26}$$

for some $k, n \in \mathbf{N}$, then we obtain

$$\begin{aligned} \ell &= \frac{1}{m}\sqrt{k(k+2n)}, \\ \lambda &= 4m^2\pi^2\frac{(k+n)^2}{k(k+2n)}, \end{aligned} \tag{2.27}$$

which coincide with the results in [3]. However, equation (2.24) may have other solutions than those given in Section 2.3, (2.27) and in [3]. For example, if we choose

$$\begin{aligned} &\left(\frac{\lambda}{2} - m^2\pi^2 - \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right)^{1/2} \ell = (k+\alpha)\pi, \\ &\left(\frac{\lambda}{2} - m^2\pi^2 + \left(\lambda\left(\frac{\lambda}{4} - m^2\pi^2\right)\right)^{1/2}\right)^{1/2} \ell = (2n+1-k-\alpha)\pi, \end{aligned}$$

we obtain

$$\begin{aligned} l^2 &= \frac{(k+\alpha)(2n+1-k-\alpha)}{m^2}, \\ \lambda &= m^2\pi^2\frac{(2n+1)^2}{(k+\alpha)(2n+1-k-\alpha)}, \end{aligned}$$

for all α satisfying equation (2.24), which reduces to

$$\begin{aligned} &2m^2\pi^2\left(1+\cos^2\alpha\pi\right) - \left(\lambda - 2m^2\pi^2\right)\sin^2\alpha\pi = 4m^2\pi^2 - \lambda\sin^2\alpha\pi \\ &= 4m^2\pi^2 - m^2\pi^2\frac{(2n+1)^2}{(k+\alpha)(2n+1-k-\alpha)}\sin^2\alpha\pi \\ &= 0, \end{aligned}$$

or equivalently,

$$\sin^2\alpha\pi = \frac{4(k+\alpha)(2n+1-k-\alpha)}{(2n+1)^2} =: f_{kn}(\alpha). \tag{2.28}$$

Equation (2.28) is independent of m and has at least one solution for $k,\ n = 1, 2, \dots,$ see e.g., Figure 2.

2.4. Path Following of the Solution Curves of $q(\lambda, \mu, \ell, m) = 0$ for $\mu \in [0, 1]$

We can calculate solutions of (2.20) either from μ =0 to μ =1 or backward from μ =1 to μ =0 by using the numerical continuation methods [9] as follows.

1. Starting with $\mu = 1$ and the solutions in (2.23), we trace the solution curve down to $\mu = 0$.
2. Start with $\mu = 0$ and fix ℓ. First, we solve (2.24) for $m = 1, 2, \dots,$ see e.g., Figure 3.

Next, we trace the solution curves until $\mu = 1$ is reached.

ALGORITHM 2. Path following for $q(\lambda, \mu, \ell, m) = 0$.

- Let $m = 1, 2, \dots$. Starting from λ_0, $\mu_0 = 0$, for $j = 1, 2, \dots, N$ do.
- Choose predictor $\tilde{\lambda}_0 = \lambda_j$, $\mu_j = \mu_{j-1} + 1/N$.
- Do corrections

$$\begin{aligned} &\tilde{\lambda}_i = \tilde{\lambda}_{i-1} - q_\lambda\left(\tilde{\lambda}_{i-1}, \mu_j, l, m\right)^{-1} q_\lambda\left(\tilde{\lambda}_{i-1}, \mu_j, l, m\right), \qquad i = 1, 2, \dots, \\ &\lambda_j = \tilde{\lambda}_K, \qquad \text{for some } K. \end{aligned}$$

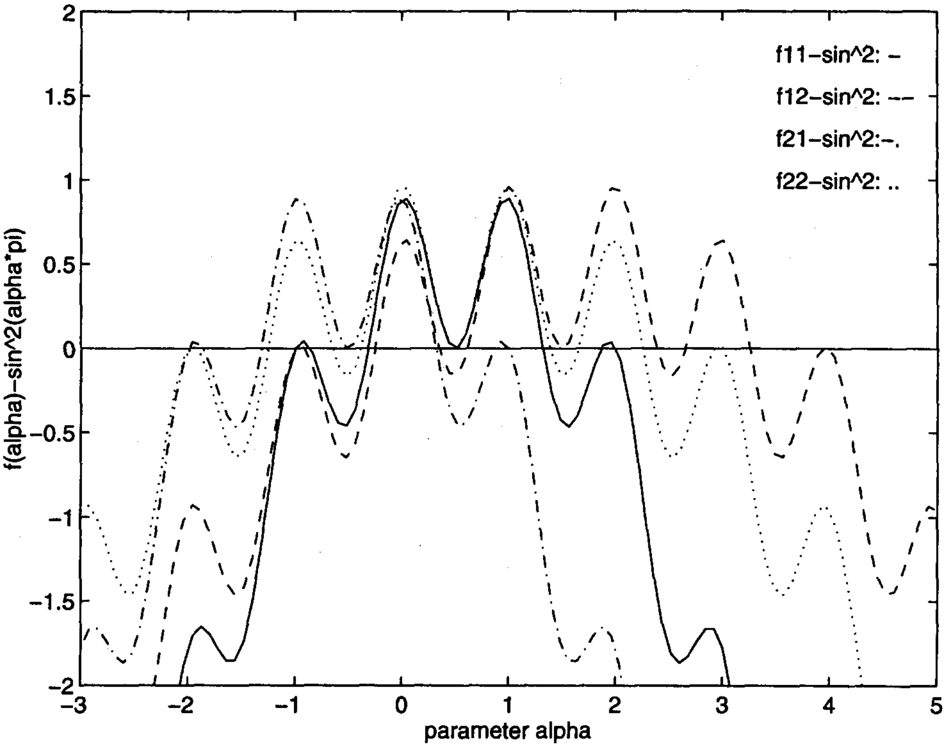

Figure 2. Possible α values given as intersections of the curves $f_{kn}(\alpha) - \sin^2 \alpha\pi$ with the horizontal axis.

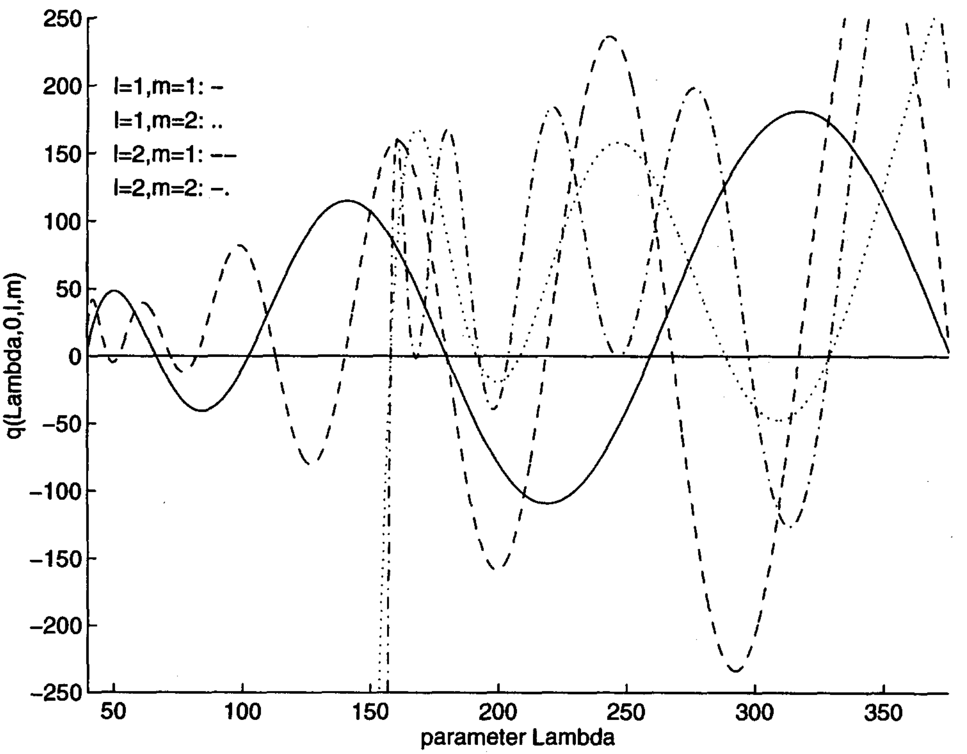

Figure 3. Curves of the functions $q(\lambda, 0, l, m)$. Intersections with the axis $q = 0$ are solutions of the equation $q(\lambda, 0, l, m) = 0$.

3. BIFURCATION ANALYSIS

Different from the variational formulations in [3], we consider directly the von Kármán equations in $C^4(\Omega)$ with the boundary conditions (1.7), which are satisfied weakly in the Sobolev spaces. We are interested in nontrivial solutions at simple bifurcation points and the phenomenon of mode jumping at the first double bifurcation point on the trivial solution curve. To this end, we consider the load λ, the length ℓ of the rectangle, and the homotopy variable μ as bifurcation parameters.

To emphasize the length ℓ of the rectangle as the second bifurcation parameter, we make it appear explicitly in the equations by the transformation $(x,y) \leftrightarrow (\ell\tilde{x}, y)$ of the domain $\Omega \leftrightarrow \tilde{\Omega} := [0,1] \times [0,1]$ as done in [3]. Then we rewrite problem (1.4) as an operator equation

$$G(w,\lambda,\ell,\mu) := w - T(\lambda,\ell,\mu)\left(w - \frac{1}{2}\left[\Delta_N^{-2}[w,w],w\right]\right) = 0, \tag{3.1}$$

where $T(\lambda,\ell,\mu) : \mathbf{C}(\Omega) \to \mathbf{C}^4(\Omega)$ is the linear operator with $T(\lambda,\ell,\mu)g := u$, defined as the solutions of the problem

$$\begin{aligned}
\frac{1}{\ell^4}\frac{\partial^4 u}{\partial x^4} + \frac{2}{\ell^2}\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{\lambda}{\ell^2}\frac{\partial^2 u}{\partial x^2} + u = g, &\qquad \text{in } \tilde{\Omega} = [0,1] \times [0,1],\\
-\frac{g_0(\mu)}{\ell}\frac{\partial u}{\partial x} + \frac{g_1(\mu)}{\ell^2}\frac{\partial^2 u}{\partial x^2} = 0, &\qquad \text{for } x = 0,\\
\frac{g_0(\mu)}{\ell}\frac{\partial u}{\partial x} + \frac{g_1(\mu)}{\ell^2}\frac{\partial^2 u}{\partial x^2} = 0, &\qquad \text{for } x = 1,\\
\frac{\partial^2 u}{\partial y^2} = 0, &\qquad \text{for } y = 0,1,\\
u = 0, &\qquad \text{on } \partial\Omega.
\end{aligned} \tag{3.2}$$

For convenience here we have replaced the variable $\tilde{x}$ by x. Note that the dependence of G on the parameter μ in the boundary conditions (1.7) is expressed implicitly in the linear operator $T(\lambda,\ell,\mu)$.

However, the homotopy parameter μ may appear explicitly in the equation if we consider the weak form of (3.1) in the Sobolev space $H^2(\Omega)$ defined by a bilinear form which includes appropriate boundary integrations to realize the boundary conditions (1.7), see e.g., [10] for semilinear second-order elliptic problems. To simplify the discussion, we avoid here the techniques involved in the justification of smoothness of solutions of (3.1) with respect to ℓ,μ and remind that it is ensured for rectangular domain with appropriate boundary conditions, see e.g., [11].

Obviously, $G(0,\lambda,\ell,\mu) = 0$, furthermore, bifurcation points along the trivial solution consist of $(0,\lambda(\ell,\mu,m))$ with $\lambda(\ell,\mu,m)$ as solutions of (2.20).

3.1. Simple Bifurcation Points

Let $m \in \mathbf{N}$ be arbitrary. For $\lambda(\ell,\mu,m) > 4m^2\pi^2$ which satisfies (2.20), Theorem 1 shows that the linearized equation (2.1) has a nontrivial solution, i.e., the associated eigenfunction

$$\phi(x,y) = (\alpha_1\psi_1(x) + \alpha_2\psi_2(x))\sin m\pi y,$$

where ψ_1, ψ_2 are defined in (2.22) and (α_1,α_2) is a solution of the linear system (2.18). Moreover, the eigenspace associated with $\lambda(\ell,\mu,m)$ is one dimensional for almost all $\ell > 0$, $\mu \in [0,1]$, i.e.,

$$\ker\left(D_w G\left(0,\lambda(\ell,\mu,m),\ell,\mu\right)\right) = \operatorname{span}[\phi].$$

In other words, $(0,\lambda(\ell,\mu,m))$ is a simple bifurcation point of the von Kármán equations. Without loss of generality we choose a point $\mu = \mu_0 \neq 0$, $\ell = \ell_0 > 0$ and study solutions of the von Kármán equations at $(0,\lambda_0)$, $\lambda_0 := \lambda(\ell_0,\mu_0,m)$ with the well-known Liapunov-Schmidt method (cf. [12]).

In the sequel, $DG_0, D_wG_0, \ldots$, will denote evaluations of the derivatives $DG(w,\lambda,\ell,\mu)$, D_wG $(w,\lambda,\ell,\mu),\ldots$, at $(0,\lambda_0,\ell_0,\mu_0)$. Consider the spaces

$$X := C_0^4(\Omega) := \left\{u \in C^{4,s}(\Omega);\ u \text{ satisfies (2.2)}\right\}, \qquad Y := C(\Omega),$$

and the $L^2(\Omega)$ product. Here $C^{k,s}(\Omega)$ is the space of k-times differentiable and Hölder continuous functions of exponent s on the closure of Ω.

Since the operator

$$D_wG_0 = I - T(\lambda_0,\ell_0,\mu_0)$$

is self-adjoint and Fredholm of index zero, we have the decompositions

$$X = \ker(D_wG_0) \oplus M, \qquad Y = N \oplus \operatorname{Im}(D_wG_0), \qquad \text{with } \dim(N) = \dim(\ker(D_wG_0)). \tag{3.3}$$

Here $\oplus$ denotes orthogonal sum with respect to the $L^2(\Omega)$ product. Therefore, we write w,λ,ℓ,μ as

$$\begin{aligned} w &= \alpha\phi + v, \qquad v \in M,\\ \lambda &= \lambda_0 + \beta,\\ \ell &= \ell_0 + \delta,\\ \mu &= \mu_0 + \nu. \end{aligned} \tag{3.4}$$

Define a projection $Q : Y \to \operatorname{Im}(D_wG_0)$ with

$$Qw := w - \alpha\langle\phi,\ w\rangle\phi, \qquad \text{for all } w \in Y.$$

We decompose equation (3.1) into two parts

$$QG(\alpha\phi + v,\ \lambda_0 + \beta,\ \ell_0 + \delta,\ \mu_0 + \nu) = 0, \tag{3.5}$$

$$\langle\phi,\ G(\alpha\phi + v,\ \lambda_0 + \beta,\ \ell_0 + \delta,\ \mu_0 + \nu)\rangle = 0. \tag{3.6}$$

Equation (3.5) is equivalent to

$$\begin{aligned} v - T_0 v &= -Qr(\alpha,\beta,v),\\ \langle\phi, v\rangle &= 0, \end{aligned} \tag{3.7}$$

with $T_0 := T(\lambda_0,\ell_0,\mu_0)$ and

$$\begin{aligned} r(\alpha,\beta,\delta,\nu,v) :&= -\left(\beta\frac{\partial T_0}{\partial\lambda} + \nu\frac{\partial T_0}{\partial\mu} + \delta\frac{\partial T_0}{\partial\ell}\right)(\alpha\phi + v)\\ &\quad + \frac{1}{2}T(\lambda,\ell,\mu)\left[\Delta_N^{-2}\left[\alpha\phi + v, \alpha\phi + v\right], \alpha\phi + v\right]\\ &\quad + O\left(\|(\beta,\delta,\nu)\|^2\right). \end{aligned}$$

This equation is uniquely solvable for v as a function of $\alpha,\ \beta,\ \delta$, and ν. Moreover,

$$v(0,0,0,0) = 0, \qquad v(\alpha_1,\beta,\delta,\nu) = O\left(\|(\alpha,\beta,\delta,\nu)\|^2\right).$$

Substituting this into (3.6) yields the reduced bifurcation equations, which are equivalent to

$$\langle\phi,\ r\left(\alpha,\beta,\delta,\nu,v\left(\alpha,\beta,\delta,\nu\right)\right)\rangle = 0. \tag{3.8}$$

Therefore, solutions of equations (3.8) have a one-to-one correspondence to those of the von Kármán equations via the solution $v(\alpha,\beta,\delta,\nu)$ of (3.7). The analysis of determinacy with multiple scalings in the singularity theory ensures that the reduced bifurcation equations of the von Kármán equations, truncated at cubic order, are sufficient for a qualitative description of the bifurcation scenario at the simple and double bifurcation points on the trivial solution curve, see [12] for more details. Taking (3.8) and the structure of v into account, we calculate the Taylor expansion of the left-hand side of (3.8) up to the cubic order

$$(a\beta\alpha + b\mu + c\delta)\alpha + d\alpha^3 = 0, \tag{3.9}$$

where

$$\begin{aligned}
a &:= -\left\langle \phi_i, \frac{\partial T_0}{\partial \lambda}\phi \right\rangle, \\
b &:= -\left\langle \phi, \frac{\partial T_0}{\partial \mu}\phi \right\rangle, \\
c &:= -\left\langle \phi, \frac{\partial T_0}{\partial \ell}\phi \right\rangle, \\
d &:= \left\langle \phi, T_0\left[\Delta_N^{-2}[\phi,\phi],\phi\right] \right\rangle.
\end{aligned}$$

From the reduced equation (3.9), we see that bifurcation of the von Kármán equations at $(0, \lambda_0)$ for $\mu = \mu_0$, $\ell = \ell_0$ is pitchfork with respect to all three parameters λ, μ, ℓ. Our numerical results in Section 5 verify this aspect.

Evidently, if the eigenfunction ϕ is known, the coefficient d can be calculated directly. However, the coefficients a, b, c involve derivatives of the operator $T(\lambda, \ell, \mu)$ with respect to λ, ℓ, μ, and need more consideration. For any arbitrary function $g \in C(\tilde{\Omega})$, let

$$u(\lambda, \ell, \mu) := T(\lambda, \ell, \mu)g.$$

According to definition (3.2) of $T(\lambda, \ell, \mu)$, the function $\frac{\partial u}{\partial \ell}(\lambda, \ell, \mu) = \frac{\partial T}{\partial \ell}(\lambda, \ell, \mu)g$ satisfies

$$\begin{aligned}
&\left(\frac{1}{\ell^4}\frac{\partial^4}{\partial x^4} + \frac{2}{\ell^2}\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} + \frac{\lambda}{\ell^2}\frac{\partial^2}{\partial x^2} + I\right)\frac{\partial u}{\partial \ell}(\lambda, \ell, \mu) \\
&= -\left(-\frac{4}{\ell^5}\frac{\partial^4}{\partial x^4} - \frac{4}{\ell^3}\frac{\partial^4}{\partial x^2 \partial y^2} - \frac{2\lambda}{\ell^3}\frac{\partial^2}{\partial x^2}\right)u(\lambda, \ell, \mu), \qquad \text{in } \tilde{\Omega} = [0,1] \times [0,1], \\
&\left(-\frac{g_0(\mu)}{\ell}\frac{\partial}{\partial x} + \frac{g_1(\mu)}{\ell^2}\frac{\partial^2}{\partial x^2}\right)\frac{\partial u}{\partial \ell}(\lambda, \ell, \mu) = \frac{g_1}{\ell^3}\frac{\partial^2 u}{\partial x^2}(\lambda, \ell, \mu), \qquad \text{for } x = 0, \\
&\left(\frac{g_0(\mu)}{\ell}\frac{\partial}{\partial x} + \frac{g_1(\mu)}{\ell^2}\frac{\partial^2}{\partial x^2}\right)\frac{\partial u}{\partial \ell}(\lambda, \ell, \mu) = \frac{g_1}{\ell^3}\frac{\partial^2 u}{\partial x^2}(\lambda, \ell, \mu), \qquad \text{for } x = 1, \\
&\frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial \ell}(\lambda, \ell, \mu) = 0, \qquad \text{for } y = 0, 1, \\
&\frac{\partial u}{\partial \ell}(\lambda, \ell, \mu) = 0, \qquad \text{on } \partial\Omega.
\end{aligned} \tag{3.10}$$

Here we have utilized the boundary conditions of $u(\ell, \mu)$. One could solve this equation numerically and then calculate the $L^2(\Omega)$-product with ϕ. On the other hand, note that in the calculations of the coefficients a, b, c in (3.10), the derivatives of $T(\lambda, \ell, \mu)$ are not used directly. Hence, let $u(\lambda, \ell, \mu) := T(\lambda, \ell, \mu)\phi$, $u_0 := u(\lambda_0, \ell_0, \mu_0) = \phi$. At $\lambda = \lambda_0$, $\ell = \ell_0$, $\mu = \mu_0$, multiplying (3.10) with ϕ and integrating over $\tilde{\Omega}$, we obtain

$$c = -\left\langle \phi, \frac{\partial T_0}{\partial \ell}\phi \right\rangle = \frac{2}{\ell_0}\int_\Omega \phi \frac{\partial^2}{\partial x^2}(2\Delta\phi + \lambda_0\phi)\,dx\,dy - \frac{1}{\ell_0}\int_0^1 \left(\frac{\partial \phi}{\partial x}\frac{\partial^2 u}{\partial x^2}\right)\,dy. \tag{3.11}$$

This result coincides with the one in [3] for simply supported and for clamped boundary conditions. The Robin boundary conditions change coefficients of the reduced bifurcation equations not only through the basis vector ϕ, but also through boundary integrations.

Similar discussions lead to

$$a = -\left\langle \phi,\ \frac{\partial T_0}{\partial \lambda}\phi \right\rangle = \left\langle \phi,\ T_0 \frac{\partial^2}{\partial x^2}(T_0\phi) \right\rangle = \left\langle \phi,\ \frac{\partial^2 \phi}{\partial x^2} \right\rangle \tag{3.12}$$

and

$$b = g_1^{-1} \int_0^1 \frac{\partial \phi}{\partial x} \left(g_0' \frac{\partial}{\partial x} - g_1' \frac{\partial^2}{\partial x^2} \right) \phi \bigg|_{x=0} dy - g_1^{-1} \int_0^1 \frac{\partial \phi}{\partial x} \left(g_0 \frac{\partial}{\partial x} + g_1 \frac{\partial^2}{\partial x^2} \right) \phi \bigg|_{x=1} dy. \tag{3.13}$$

3.2. Double Bifurcation Points

Assume that $\lambda_m^1(\ell,\mu)$ and $\lambda_k^2(\ell,\mu)$ are two different solution curves of (2.20). The corresponding nontrivial solutions of the linearized von Kármán equation (2.1) are

$$\phi_1(x,y) = \left(\alpha_1^1\psi_1^1(x) + \alpha_2^1\psi_2^1(x)\right)\sin m\pi y, \qquad \phi_2(x,y) = \left(\alpha_1^2\psi_1^2(x) + \alpha_2^2\psi_2^2(x)\right)\sin k\pi y,$$

where (α_1^i, α_2^i), $i = 1,2$ satisfy (2.18) and ψ_1^i, ψ_2^i, $i = 1,2$ are defined by (2.22) corresponding to λ_m^1 and λ_k^2, respectively.

We consider both the homotopy parameter μ and length ℓ as bifurcation parameters. Suppose that at some point $\ell = \ell_0$, $\mu = \mu_0$ the equality

$$\lambda_m^1(\ell_0,\mu_0) = \lambda_k^2(\ell_0,\mu_0) =: \lambda_0$$

holds. If $m \neq k$, the functions ϕ_1 and ϕ_2 are linearly independent and $(0,\lambda_0)$ is a double bifurcation point of (1.1) with

$$\ker\left(\Delta^2 + \lambda_0 \frac{\partial^2}{\partial x^2}\right) = \mathrm{span}\,[\phi_1,\phi_2].$$

If $m = k$, and ϕ_1, ϕ_2 are still linearly independent, then the coefficient matrix of Eq-alpha12 has two independent null vectors (α_1^i, α_2^i), $i = 1,2$, and thus must vanish. In this situation, $(0,\lambda_0)$ is a still double bifurcation point of (1.1). Moreover, ϕ_1 and ϕ_2 can be chosen as

$$\begin{aligned} \phi_1 &:= \left[g_0(\mu)\omega_2\left(\cos\omega_1 x - \cos\omega_2 x\right) + g_1(\mu)\left(\omega_1^2 - \omega_2^2\right)\sin\omega_2 x\right]\sin m\pi y, \\ \phi_2 &:= g_0(\mu)\left(\omega_2\sin\omega_1 x - \omega_1\sin\omega_2 x\right)\sin m\pi y. \end{aligned} \tag{3.14}$$

For $g_1(0) = 0$, i.e., the clamped edges, the elements ϕ_1, ϕ_2 in (3.14) reduce to those in [3]. Similarly, for the simply supported boundary conditions, i.e., $g_0(1) = 0$, statement (3.14) coincides with the classical definitions, see also [3].

Suppose that $(0,\lambda_0)$ is a double bifurcation point of the von Kármán equations for $\ell = \ell_0$ and $\mu = 0$, i.e., the partially clamped boundary conditions. Fixing ℓ and varying the homotopy parameter μ, we shift the boundary conditions from partially clamped to simply supported. Generically, the double bifurcation $(0,\lambda_0)$ persists along the homotopy, namely, dimension of the nullspace $\ker(\Delta^2 + \lambda_0 \frac{\partial^2}{\partial x^2})$ remains to be two for all $\mu \in [0,1]$. Bifurcation analysis at a double bifurcation point of the von Kármán equations with Robin boundary conditions (1.7) is lengthy and will be discussed elsewhere.

4. CENTRAL DIFFERENCE APPROXIMATIONS FOR THE VON KÁRMÁN EQUATIONS

In this section, we consider the central difference approximations of (1.1) with the Robin boundary conditions (1.7).

4.1. Discretization for the Linearized von Kármán Equation

To start with, we consider the linearized von Kármán equation

$$\begin{aligned} \Delta^2 w + \lambda \frac{\partial^2 w}{\partial x^2} &= 0, && \text{in } \Omega = [0,\ell] \times [0,1], \\ w = g_0(\mu)\frac{\partial w}{\partial n} + g_1(\mu)\Delta w &= 0, && \text{on } x = 0, \ell, \\ w = \Delta w &= 0, && \text{on } y = 0, 1. \end{aligned} \tag{4.1}$$

We discretize (4.1) by the 13-point central difference approximations with uniform meshsize $h = 1/(L+1)$ on the x- and y-axis, respectively. For convenience we assume that ℓ is divisible by h, say $\ell/h = K+1$ for some positive integer K. Let $W(x_i, y_j) = W_{i,j}$ be the net function defined on the set of mesh points

$$\{(x_i, y_j) \mid 0 \le i \le K+1,\ 0 \le j \le L+1\}, \qquad \text{where } K = \ell(L+1) - 1.$$

At the discrete points the function $w(x,y)$ is approximated by a net function $W(x_i, y_j)$. The boundary conditions on $y = 0$ and $y = 1$ imply that

$$W_{i,j+1} = -W_{i,j-1}, \qquad j = 0,\ L+1,$$

while the boundary conditions on $x = 0$ and $x = \ell$ imply that

$$\begin{aligned} -g_0(\mu)\frac{W_{1,j} - W_{1,j}}{2h} + g_1(\mu)\frac{W_{1,j} - 2W_{0,j} + W_{1,j}}{h^2} + O\left(h^2\right) &= 0, \\ g_0(\mu)\frac{W_{K+2,j} - W_{K,j}}{2h} + g_1(\mu)\frac{W_{K+2,j} - 2W_{K+1,j} + W_{K,j}}{h^2} + O\left(h^2\right) &= 0. \end{aligned}$$

Neglecting the discretization errors and noting that $W_{i,j} = 0$, on $\partial\Omega$, we have

$$\begin{aligned} W_{-1,j} &= \frac{g_0(\mu)h - 2g_1(\mu)}{g_0(\mu)h + 2g_1(\mu)} W_{1,j}, \\ W_{K+2,j} &= \frac{g_0(\mu)h - 2g_1(\mu)}{g_0(\mu)h + 2g_1(\mu)} W_{K,j}. \end{aligned}$$

Thus, we obtain the discretization matrix A associated to the operator Δ^2

$$A = \frac{1}{h^4} \begin{pmatrix} A_{K_1} & P_K & I_K & & & & \\ P_K & A_{K_2} & P_K & I_K & & & \\ I_K & P_K & A_{K_3} & P_K & I_K & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & I_K \\ & & & I_K & P_K & A_{K_{L-1}} & P_K \\ & & & & I_K & P_K & A_{K_L} \end{pmatrix} \in \mathbf{R}^{KL \times KL}, \tag{4.2}$$

where

$$A_{K_i} = \begin{pmatrix} a_i + \delta & -8 & 1 & & & & \\ -8 & a_i & -8 & 1 & & & \\ 1 & -8 & a_i & -8 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -8 & a_i & -8 & 1 \\ & & & 1 & -8 & a_i & -8 \\ & & & & 1 & -8 & a_i + \delta \end{pmatrix}$$

with

$$\delta = \frac{g_0(\mu)h - 2g_1(\mu)}{g_0(\mu)h + 2g_1(\mu)} \quad \text{and} \quad a_i := \begin{cases} 19, & i = 1,\ L, \\ 20, & i = 2, \dots, L-1, \end{cases} \tag{4.3}$$

and

$$P_K = \begin{pmatrix} -8 & 2 & & & 0 \\ 2 & -8 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & -8 & 2 \\ 0 & & & 2 & -8 \end{pmatrix} \in \mathbf{R}^{K\times K}.$$

Let D be discretization matrix corresponding to the differential operator $\frac{\partial^2}{\partial x^2}$ defined on $[0,\ell] \times [0,1]$. Then

$$D = \frac{1}{h^2}\operatorname{diag}(C_K, \dots, C_K) \in \mathbf{R}^{KL\times KL} \tag{4.4}$$

with

$$C_K = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \in \mathbf{R}^{K\times K}.$$

The central difference analogue of (4.1) can be written as

$$H(W, \lambda) = AW + \lambda DW = 0. \tag{4.5}$$

Let $\bar{A}$ be the discretization matrix corresponding to the operator Δ^2 defined on $[0,\ell] \times [0,1]$ with simply supported boundary conditions. Note that $\bar{A}$ is positive definite, see [13]. Let $\bar{D} := \operatorname{diag}(1+\delta, 0, \dots, 0, 1+\delta) \in \mathbf{R}^{K\times K}$, we have $A = \bar{A} + I_L \otimes \bar{D}$. By definition (4.3) of δ, for $g_0(\mu) \geq 0$ and $g_1(\mu) \geq 0$ we have

$$-1 = \frac{-g_0(\mu)h - 2g_1(\mu)}{g_0(\mu) + 2g_1(\mu)} \leq \frac{g_0(\mu)h - 2g_1(\mu)}{g_0(\mu)h + 2g_1(\mu)} \leq \frac{g_0(\mu)h + 2g_1(\mu)}{g_0(\mu)h + 2g_1(\mu)} = 1.$$

Thus, $|\delta| \leq 1$. More precisely, we consider the following two cases.

CASE 1. If $\delta = -1$, then $\bar{D} = 0$ and $A = \bar{A}$ which is symmetric and positive definite.

CASE 2. If $-1 < \delta \leq 1$, then we note that $I_L \otimes \bar{D}$ is a diagonal and positive definite matrix. It is obvious that A is symmetric and positive definite.

4.2. Discretization for the Nonlinear von Kármán Equations

For $\Omega = [0,\ell] \times [0,1]$, we discretize the von Kármán equations (1.1) with Robin boundary conditions (1.7).

Denote the discretization matrices corresponding to $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial x \partial y}$ by E and V, respectively:

$$E = \frac{1}{h^2}\begin{pmatrix} -2I_K & I_k & & & \\ I_K & -2I_K & I_K & & \\ & \ddots & \ddots & \ddots & \\ & & I_K & -2I_K & I_K \\ & & & I_K & -2I_K \end{pmatrix} \in \mathbf{R}^{KL\times KL},$$

which can be expressed as $E = h^{-2} C_L \otimes I_K$, and

$$V = \frac{1}{h^2}\begin{pmatrix} 0 & V_K & & & \\ -V_K & 0 & V_K & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & V_K \\ & & & -V_K & 0 \end{pmatrix} \in \mathbf{R}^{KL\times KL}, \tag{4.6}$$

where

$$V_K = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 0 \end{pmatrix} \in \mathbf{R}^{K\times K}.$$

We have $V = h^{-2} V_L \otimes V_K$.

For convenience we will define a vector-vector multiplication and a matrix- vector multiplication as follows. The operation will be denoted by " $*$ ".

DEFINITION 1. *For any* $\mathbf{x} = (x_1, \dots, x_N)^\top$, $\mathbf{y} = (y_1, \dots, y_N)^\top \in \mathbf{R}^N$, *we define* $\mathbf{x} * \mathbf{y} \in \mathbf{R}^N$ *by* $\mathbf{x} * \mathbf{y} = (x_1 y_1, \dots, x_N y_N)^\top$. *For any* $A = (a_1, \dots, a_N)^\top$, *where* $a_i^\top$ *denotes the* i^{th} *row of* A, *and* $\mathbf{x} = (x_1, \dots, x_N)^\top \in \mathbf{R}^N$, *we define* $A * \mathbf{x} \in \mathbf{R}^{N\times N}$ *by* $A * \mathbf{x} = (x_1 a_1, \dots, x_N a_N)^\top$.

With $Z = (F,\ W)^\top \in \mathbf{R}^{2KL}$, the discrete analogue of the von Kármán equations with Robin boundary conditions is

$$H(Z,\lambda) = (H_1(Z,\lambda),\ H_2(Z,\lambda))^\top = 0, \tag{4.7}$$

where

$$H_1(Z,\lambda) := BF + (DW) * (EW) - \frac{1}{16}(VW) * (VW),$$
$$H_2(Z,\lambda) := AW - (DW) * (EF) - (EW) * (DF) + \frac{1}{8}(VW) * (VF) + \lambda DW.$$

Here A is defined as in (4.2), and B has the same form as A except that we use discretization of the boundary conditions

$$\frac{\partial f}{\partial n} = \frac{\partial(\Delta f)}{\partial n} = 0, \qquad \text{on } \partial\Omega.$$

The Jacobian matrix corresponding to (4.7) is

$$\begin{aligned} DH(Z,\lambda) &= (D_Z H(Z,\lambda),\ D_\lambda H(Z,\lambda)) \\ &= \begin{pmatrix} B & M(Z) & DW \\ & & \\ -M(Z) & \tilde{B}(Z,\lambda) & 0 \end{pmatrix} \in \mathbf{R}^{2KL\times(2KL+1)}, \end{aligned} \tag{4.8}$$

where

$$M(Z,\lambda) := D * (EW) + E * (DW) - \frac{1}{8} V * (VW)$$

and

$$\tilde{B}(Z,\lambda) := A - D * (EF) - E * (DF) + \frac{1}{8} V * (VF) + \lambda D.$$

On the trivial solution curve $F \equiv 0,\ W \equiv 0$ the Jacobian matrix DH in (4.8) reduces to

$$DH(0,\lambda) = \begin{pmatrix} B & 0 & 0 \\ & & \\ 0 & A + \lambda D & 0 \end{pmatrix}. \tag{4.9}$$

Since B is symmetric and positive definite, the singularity of DH is determined by $A + \lambda D$, which is just the coefficient matrix in (4.5).

In the context of central difference approximations, the techniques in [14] can be adapted to perform error analysis for the discrete solution branches of (4.7), see [13] for more details.

For branch switching at simple bifurcation point we take local perturbation in the context of numerical continuation of solution curves. Suppose that $y^* = (z^*, \lambda^*)$ is a detected bifurcation point on the curve $c \in H^{-1}(0)$. Let $V \in R^{N+1}$ be a bounded open neighborhood of y^*. Let

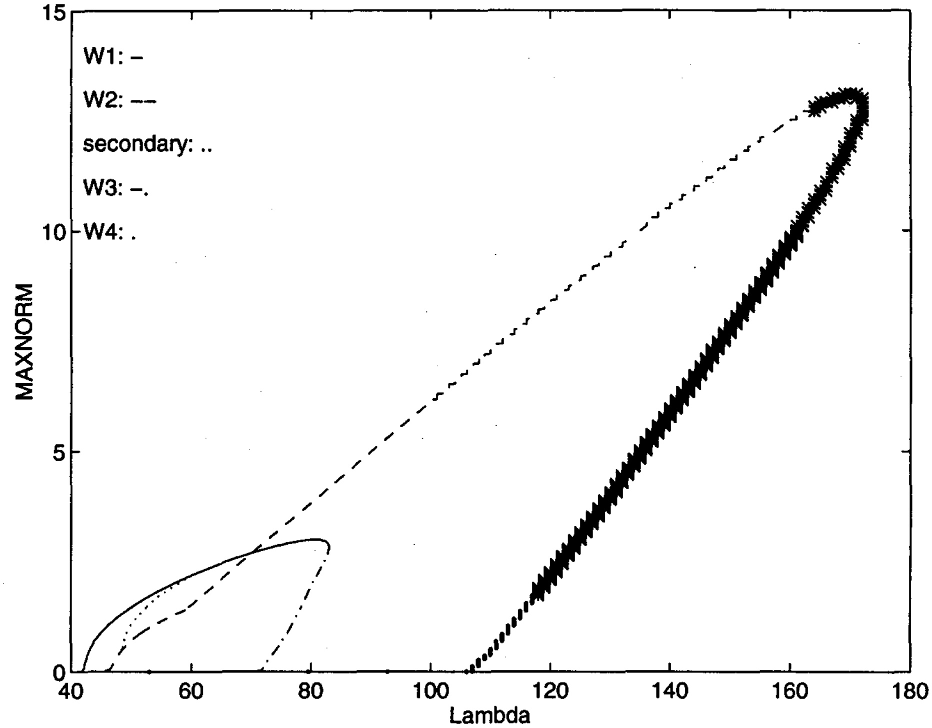

Figure 4. The first four solution curves in Example 1.

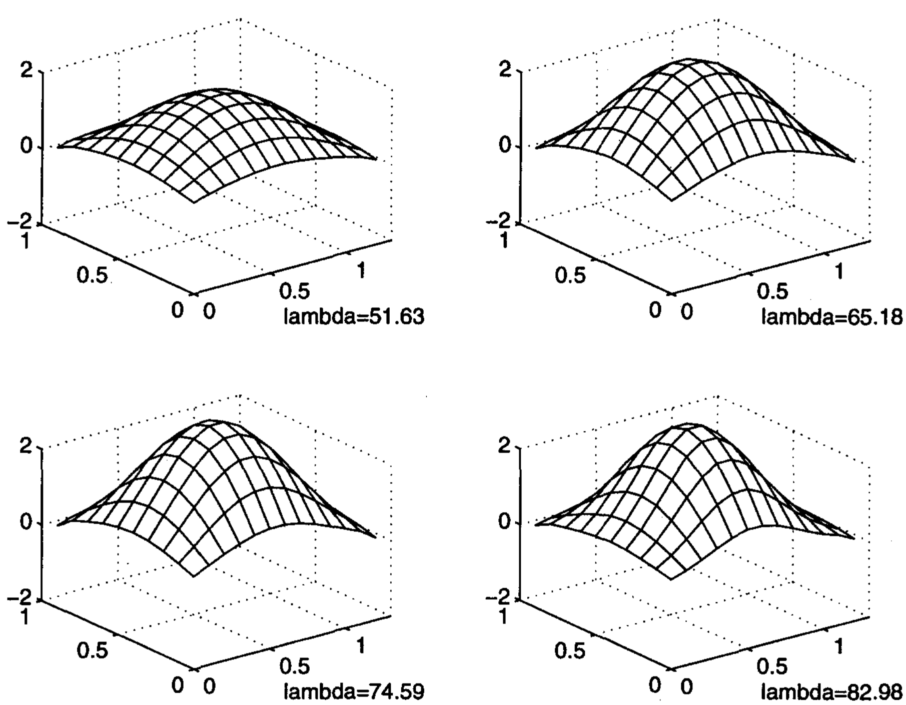

Figure 5. The contours of the $W1$ solution branch in Example 1.

$f : \mathbf{R}^{N+1} \to \mathbf{R}$ be a smooth mapping such that $f(y) = 0$, for $y \notin V$ and $f(y) > 0$ for $y \in V$ Instead of solving $H(y) = 0$, we trace the solution curves of the perturbed problem

$$H_d(y) = H(y) + f(y) \cdot d, \qquad y = (z,\ \lambda).$$

We refer to [13,15] for details and other branch-switching techniques.

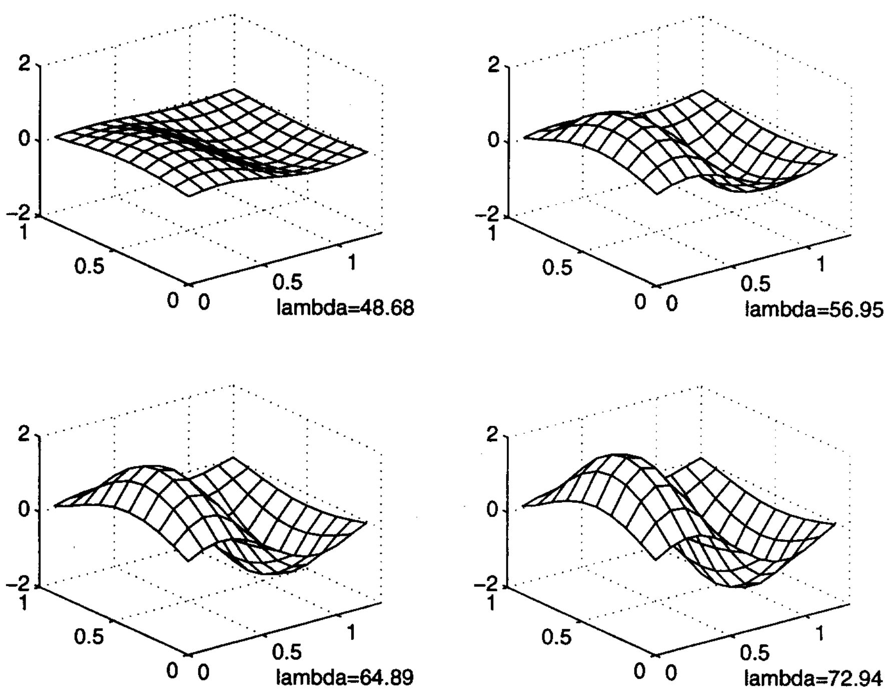

Figure 6. The contours of the $W2$ solution branch in Example 1.

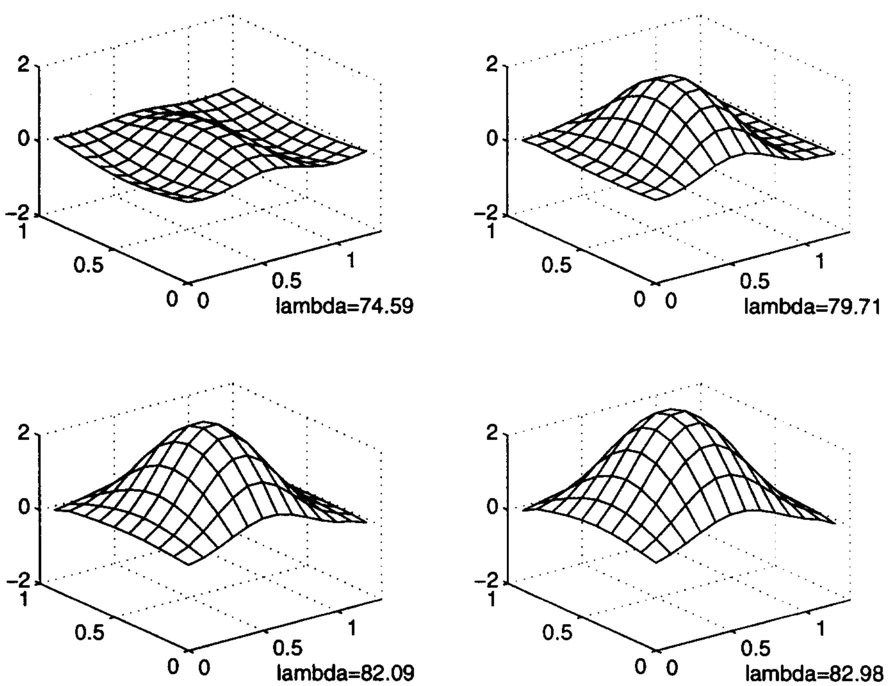

Figure 7. The contours of the $W3$ solution branch in Example 1.

5. NUMERICAL RESULTS

We start from the first two simple bifurcation points on the trivial solution curve of the von Kármán equations with Robin boundary conditions, which are obtained by the splitting of the first double bifurcation point via domain perturbation. We wish to trace the bifurcating solution branches by using the numerical continuation methods [9], where the direct method is used as

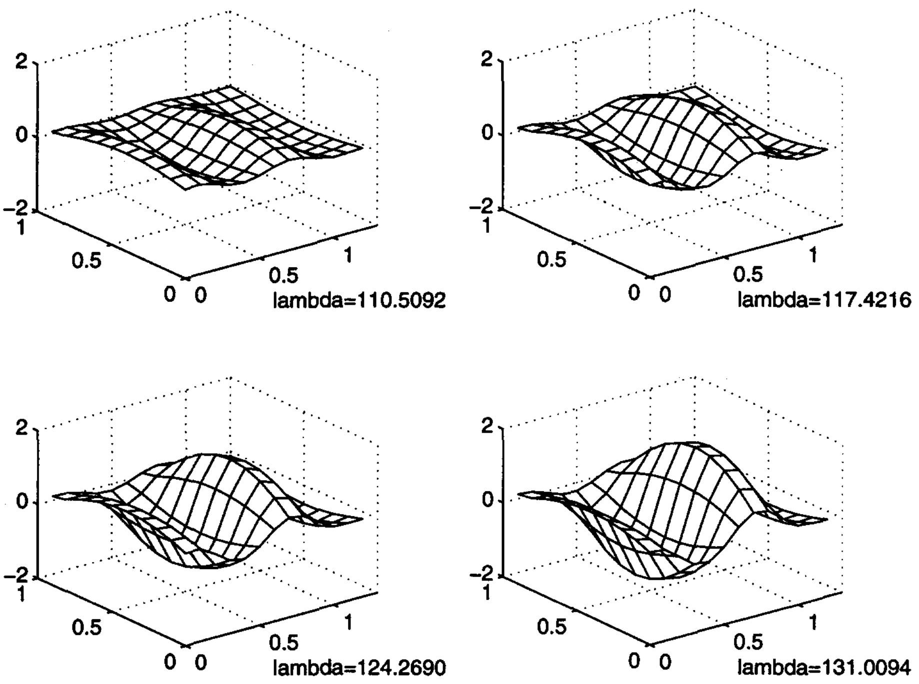

Figure 8. The contours of the $W4$ solution branch in Example 1.

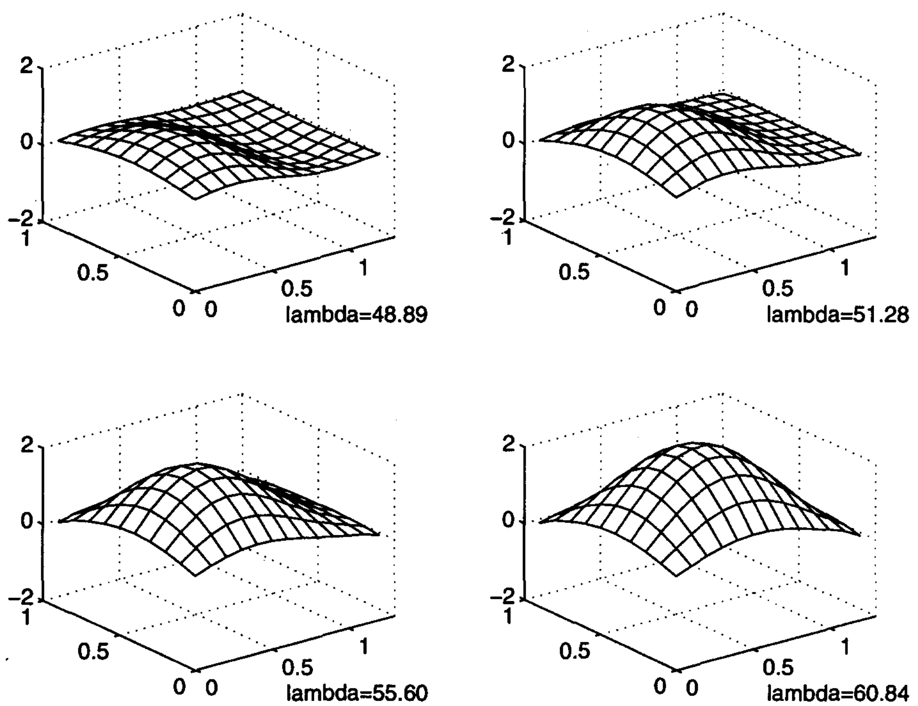

Figure 9. The contours of the secondary solution branch in Example 1.

the linear solver, see [13]. In particular, we are interested in mode jumping of the von Kármán equations, namely, secondary bifurcations which connect solution branches bifurcating from these two bifurcation points on the trivial solution curve. Throughout our numerical experiments, the stopping criterion for the Newton corrector is $5 * 10^{-4}$. The computations were performed on an IBM SP2 computer at National Chung-Hsing University.

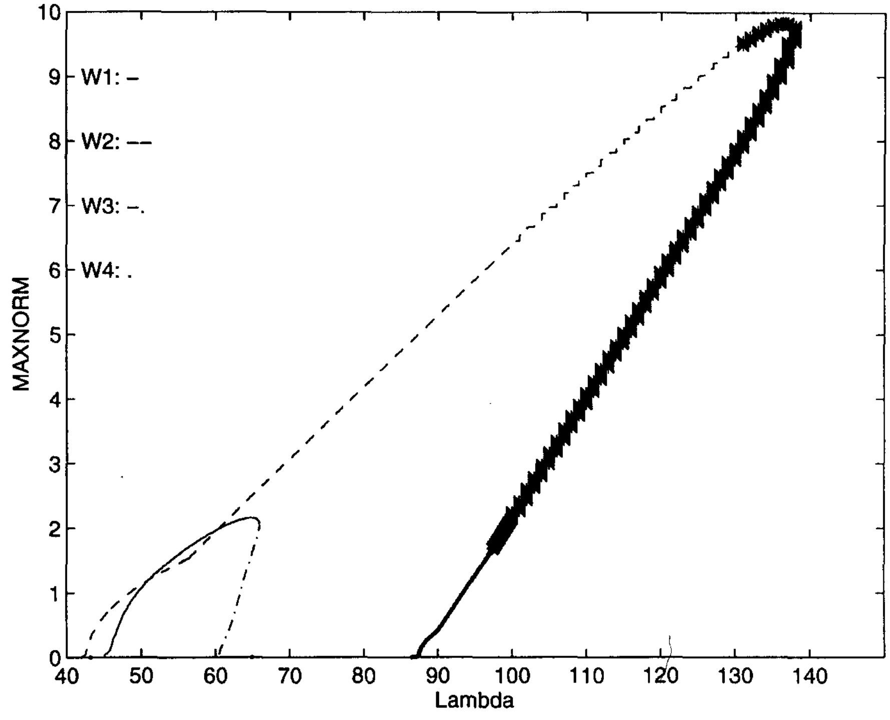

Figure 10. The first four solution branches in Example 2.

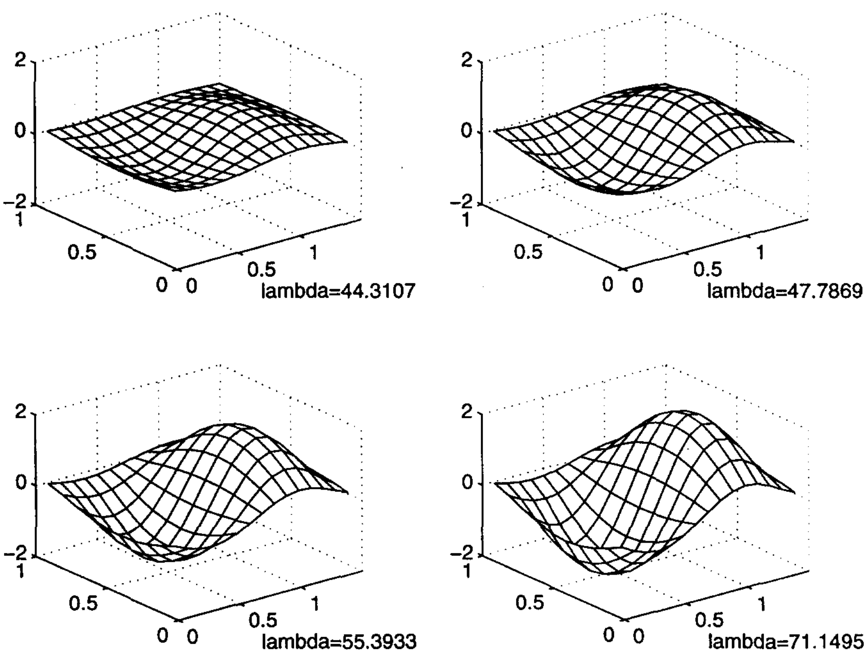

Figure 11. The contours of the $W1$ solution branch in Example 2.

In our numerical experiments, we chose the following Robin boundary conditions:

$$\begin{aligned} w = (1-\mu)\frac{\partial w}{\partial n} + \mu \Delta w &= 0, && \text{on } x = 0 \text{ and } x = \ell, \\ w = \Delta w &= 0, && \text{on } y = 0 \text{ and } y = 1, \\ \frac{\partial f}{\partial n} = \frac{\partial(\Delta f)}{\partial n} &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

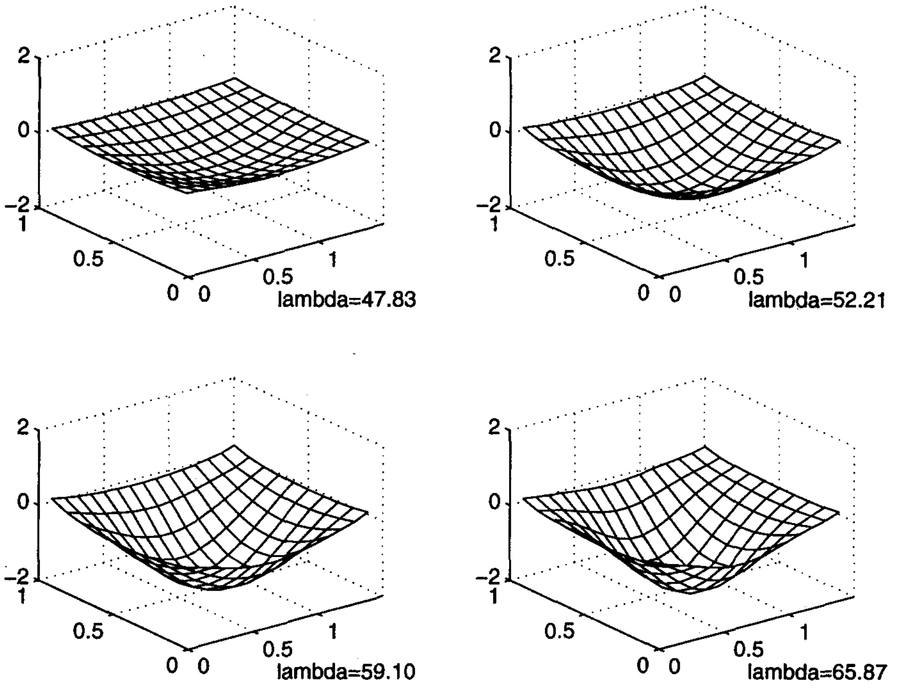

Figure 12. The contours of the $W2$ solution branch in Example 2.

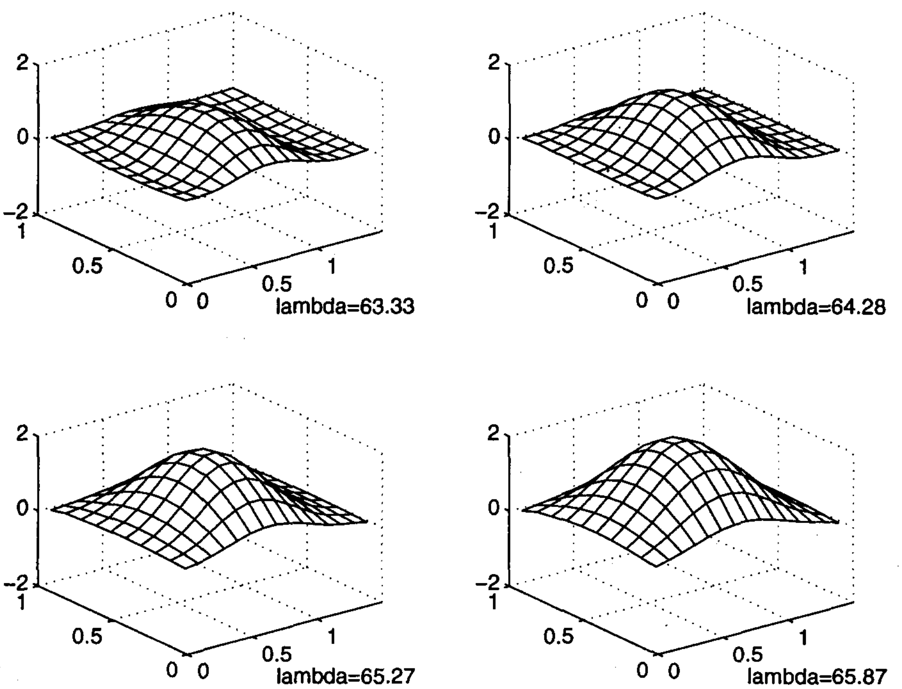

Figure 13. The contours of the $W3$ solution branch in Example 2.

The von Kármán equations are discretized by the central difference approximations with uniform meshsize h. We choose $h = 0.1$ because of the limitation of our computer facilities.

EXAMPLE 1. $\mu = 0.9$ and $\ell = 1.3$. To study whether mode jumping occurs, we perform local perturbation and domain perturbation simultaneously with $\ell = 1.3$. Figure 4 shows that the first four solution curves of the von Kármán equations, and the solution curves bifurcating from

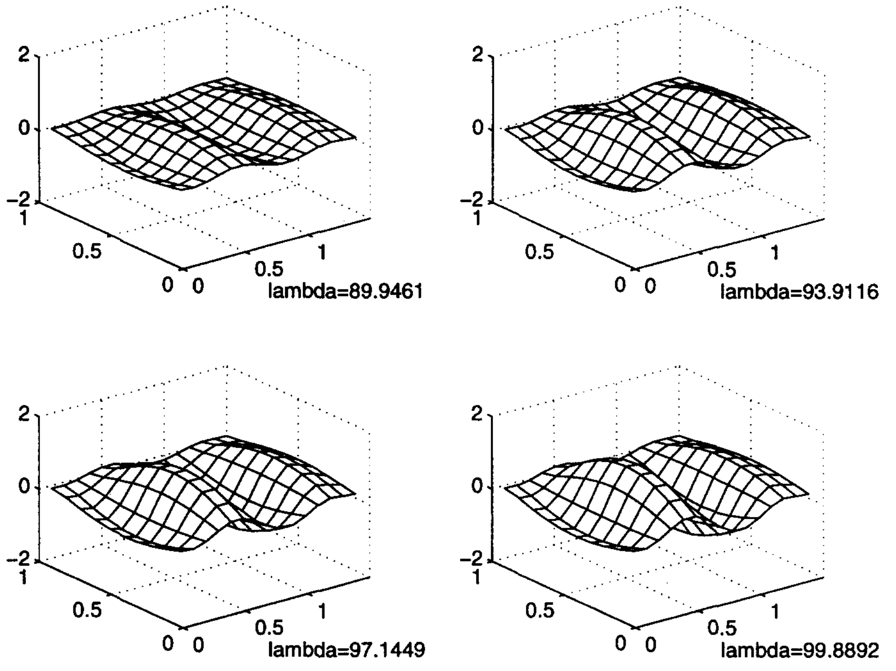

Figure 14. The contours of the $W4$ solution branch in Example 2.

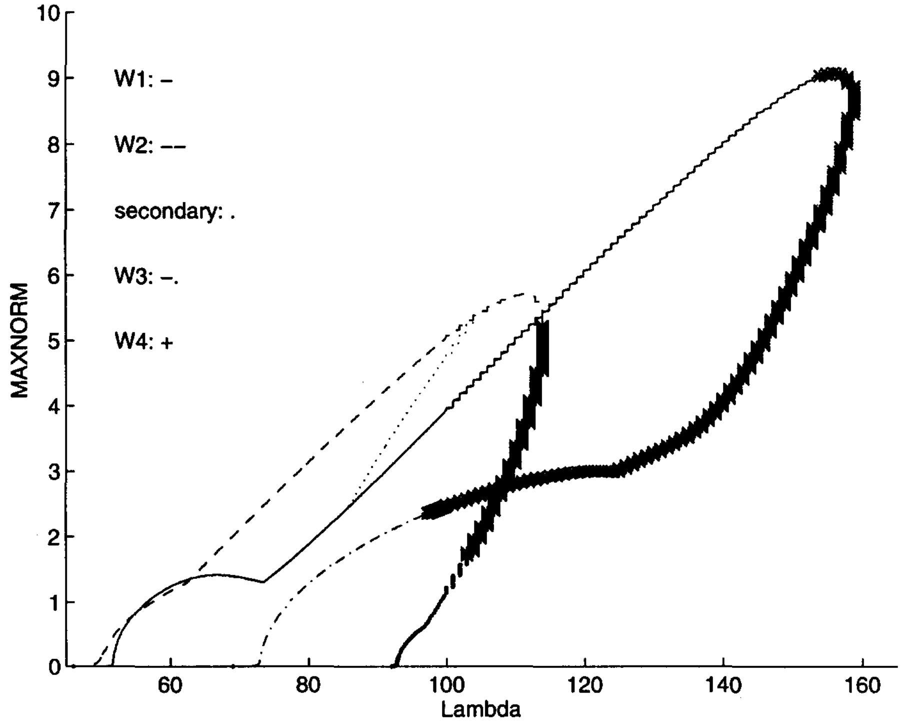

Figure 15. The first four solution curves in Example 3, $\ell = 1.6$.

$(0, \lambda_1) \approx (0, 42.24)$ and $(0, \lambda_3) \approx (0, 72.08)$ are connected with each other. Note that two secondary bifurcation points are detected at $\lambda \approx 61.0390$ and $\lambda \approx 48.9517$ on the solution curve branching from $(0, \lambda_1)$ and $(0, \lambda_2) \approx (0, 47.02)$, respectively. These two secondary bifurcation points are connected by a secondary solution branch. Thus, mode jumping occurs in this physical system. Moreover, the $W1$, $W3$ and the $W2$, $W4$ solution branches are connected with each

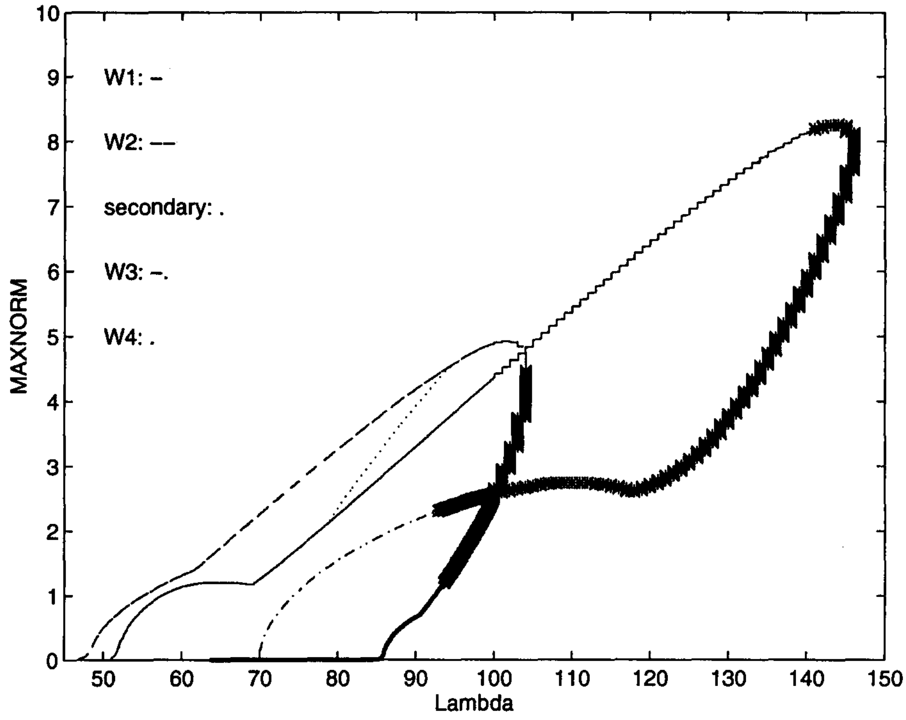

Figure 16. The first four solution curves in Example 3, $\ell = 1.7$.

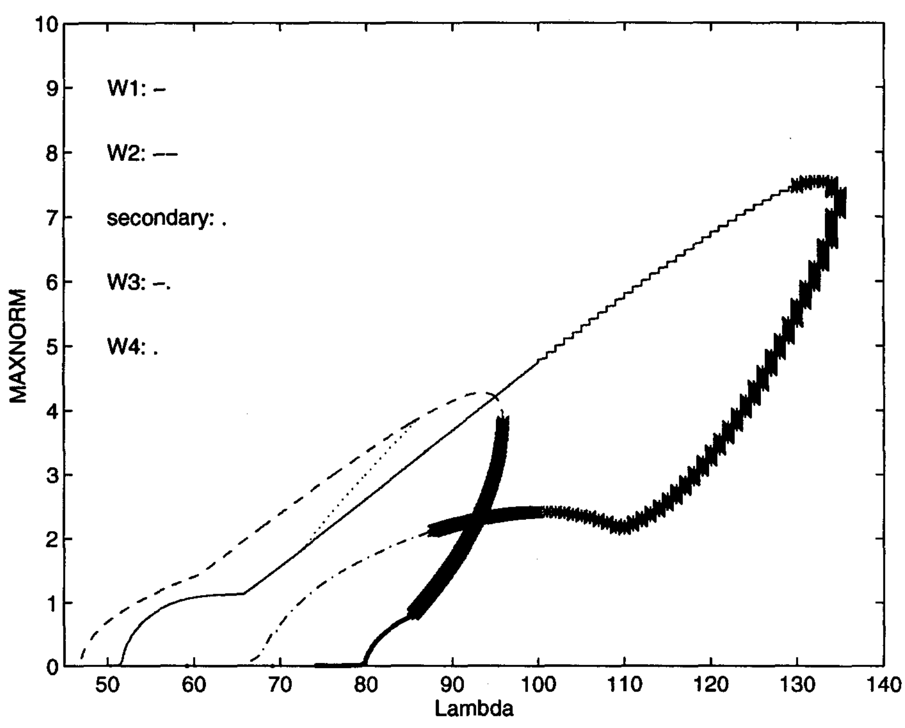

Figure 17. The first four solution curves in Example 3, $\ell = 1.8$.

other, respectively. Specifically, our numerical result shows that the $W4$ solution branch has oscillating behavior. Figures 5–8 show that the contours of the solution curves branching from $(0, \lambda_1)$, $(0, \lambda_2)$, $(0, \lambda_3)$, and $(0, \lambda_4) \approx (0, 107.35)$ for different value of λ. In Figure 9, we observe that the contour of the secondary solution branch vary with respect to λ.

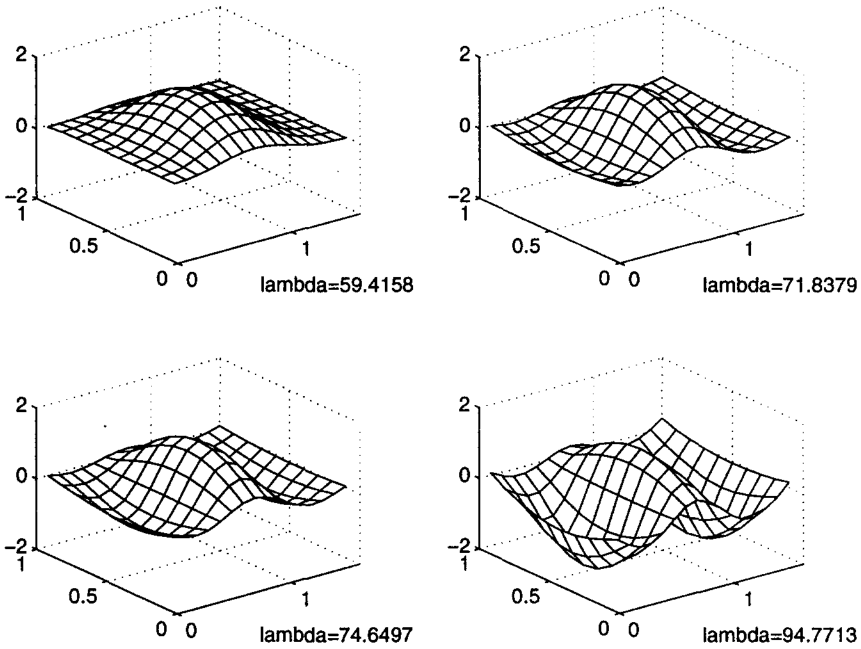

Figure 18. Contours of the $W1$ solution branch in Example 3.

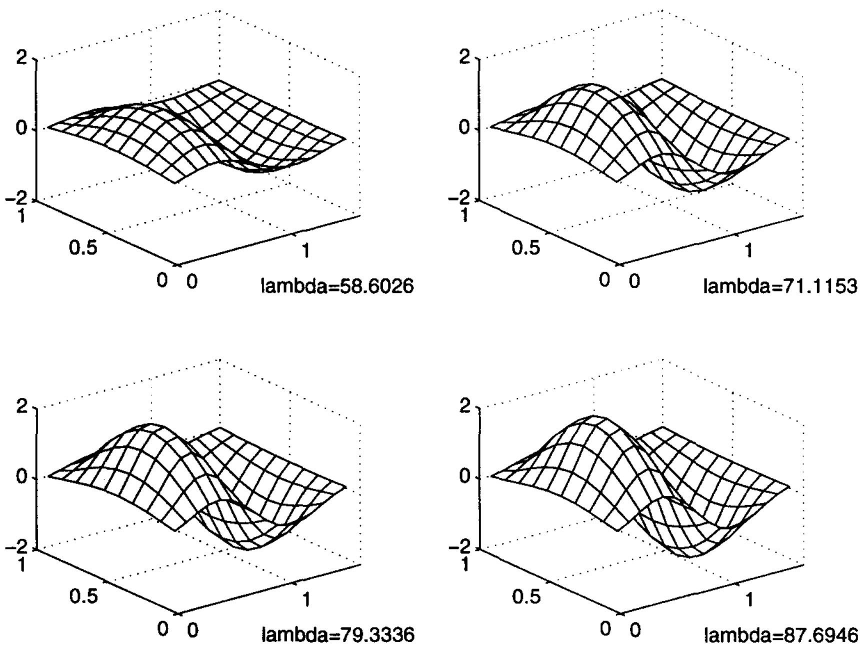

Figure 19. Contours of the $W2$ solution branch in Example 3.

EXAMPLE 2. $\mu = 0.9$ and $\ell = 1.5$. Figure 10 shows the nontrivial solution branches from $(0, \lambda_1) \approx (0, 42.72)$, $(0, \lambda_2) \approx (0, 48.18)$, and $(0, \lambda_3) \approx (0, 60.7)$, and $(0, \lambda_4) \approx (0, 87.6)$. Our numerical output shows that there is no secondary bifurcation point on the first two solution curves. Thus, mode jumping does not occur in this system. Note that the $W2$, $W3$ and $W1$, $W4$ solution branches are connected with each other. The contours for the first four solution branches are given in Figures 11–14.

EXAMPLE 3. We consider $\mu = 0.1$ with different lengths $\ell = 1.6$, 1.7, and 1.8, respectively. Figures 15–19 show that the bifurcation scenario of these three systems are very similar. In

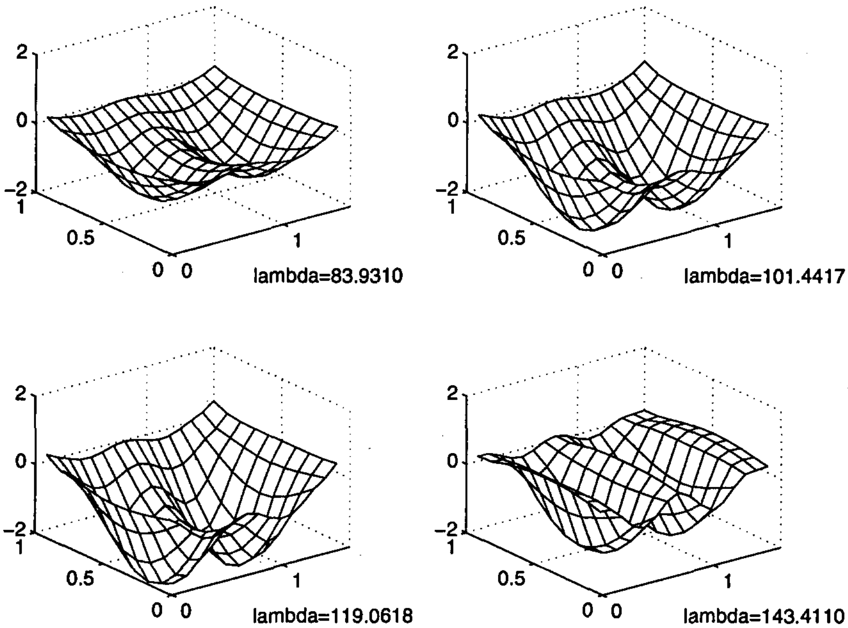

Figure 20. Contours of the $W3$ solution branch in Example 3.

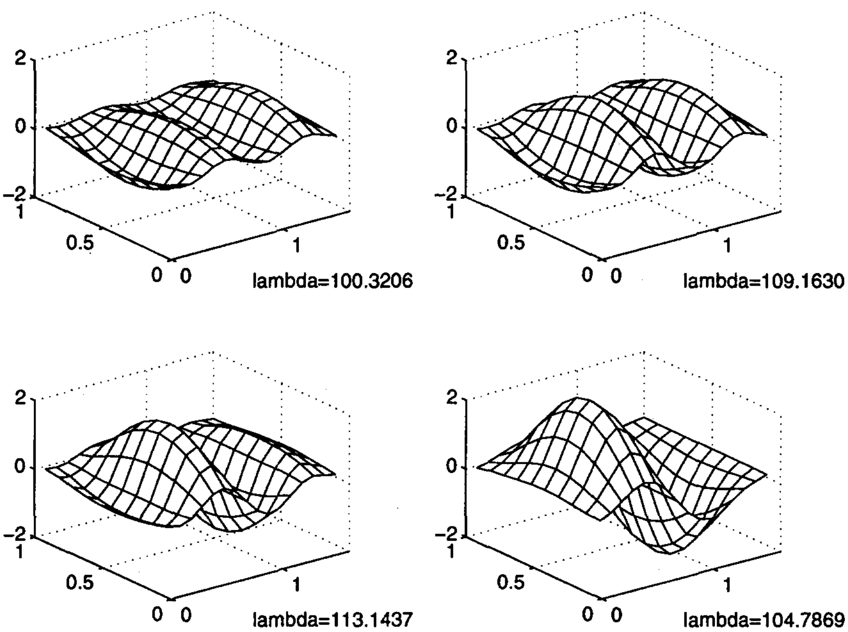

Figure 21. Contours of the $W4$ solution branch in Example 3.

particular, mode jumping occur in each of these three systems. For example, in Figure 15 a secondary bifurcation points is detected on the $W1$ solution branch for $\lambda \in (86.0928, 86.1716)$, and on the $W2$ solution branch for $\lambda \in (104.1482, 104.2391)$. These two points are connected by a secondary solution branch. Figures 18–21 show the contours of the first four solution branches in Figure 15. The contours of the secondary solution branch in this system are shown in Figure 22.

6. DISCUSSION

If we impose simply supported and partially clamped boundary conditions on the von Kármán equations defined on a rectangular domain, respectively, then mode jumping occurs in the latter system but not in the former, see e.g., [3]. We have connected these two different boundary

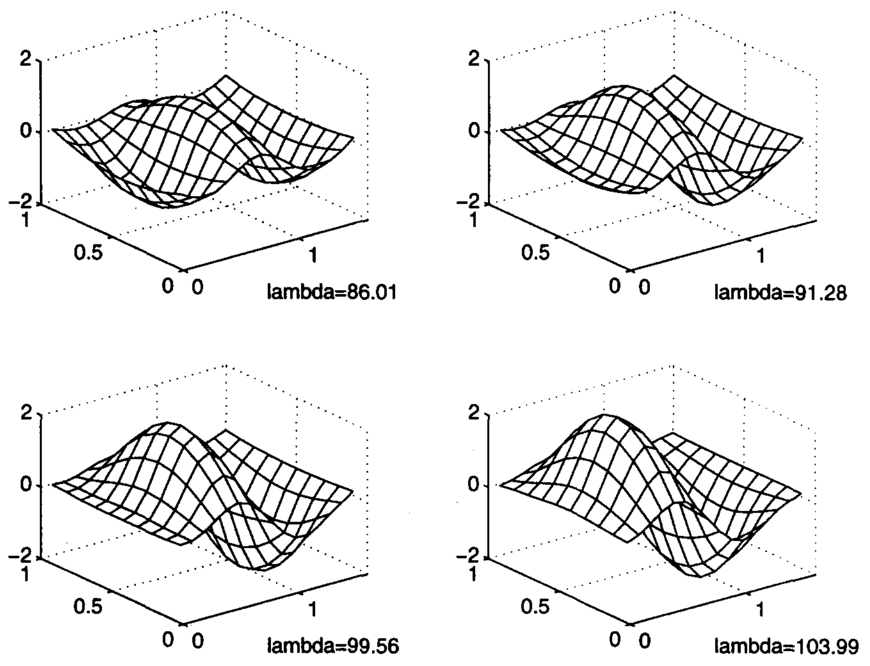

Figure 22. Contours of the secondary solution branch in Example 3.

conditions with a homotopy path, i.e., the Robin boundary conditions, investigating how mode jumping depends on the length of the domain and the homotopy parameter which corresponds to the torsional rigidity. The numerical results show that mode jumping may occur in the system. Moreover, this phenomenon is independent of the length of the domain if the boundary conditions are close to the partially clamped case. However, if the boundary conditions are close to the simply supported case, mode jumping is sensitive to the length of the domain.

Actually, there are various ways to formulate the Robin boundary conditions. For example, in the context of mixed finite element schemes, the totally clamped boundary conditions

$$f = \frac{\partial f}{\partial n} = 0, \qquad w = \frac{\partial w}{\partial n} = 0, \tag{6.1}$$

are often used, see e.g., [12; 16, Chapter 25, pp. 290–291]. In this case, we may consider the system with the Robin boundary conditions which connect (6.1) with the simply supported or the partially clamped boundary conditions, respectively. It would be interesting to investigate mode jumping of the system with different Robin boundary conditions.

In this paper, we did not discuss how to handle linear systems associated to the discrete problem. In practice, either direct methods or iterative methods can be used to solve the discretized linear systems. For example, the direct solver was used in this paper and in [13,15], while a robust continuation-unsymmetric Lanczos algorithm was proposed in [7] to trace solution curves of certain bifurcation problems. Note that an efficient linear solver for bifurcation problems should be used to detect ill conditioning of the discretization matrices as well. Recently, Brown and Walker [17] show that the GMRES [18] can be exploited to efficiently and reliably detect singularity or ill conditioning of the coefficient matrix of a linear system. We believe that it is possible to develop a more reliable and stable continuation—GMRES algorithm for bifurcation problems.

Finally, we note that in discretization of the von Kármán equations (1.1) the coefficient matrices of the linearized problems are nonsymmetric. On the other hand, by multiplying the factor -1 to one of the equations in (1.1) we obtain the self-adjointness of the linearized problem. Nevertheless, eigenvalues of the associated linear operator scatter over the whole real axis. Thus we cannot apply the iterative methods directly to solve the associated linear problems. This aspect and details of a continuation—GMRES algorithm will be discussed elsewhere.

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